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An Introduction To Analytic Platforms For Inertial Guidance

PREPARED BY
ADVANCED STUDIES GROUP
AUBURN UNIVERSITY
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AN INTRODUCTION TO ANALYTIC
PLATFORMS FOR INERTIAL NAVIGATION

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LIST OF SYMBOLS

SYMBOL	DEFINITION
X_V Y_V Z_V	Vehicle-fixed coordinate system having the orientation as shown in Appendix D.
X_S Y_S Z_S	Space-fixed coordinate system having the orientation as shown in Appendix D.
X_I Y_I Z_I	Intermediate coordinate system having the same orientation as the space-fixed system.
X' Y' Z' X'' Y'' Z''	Auxiliary coordinate systems necessary to the derivation of some of the transformations.
\vec{r}_S \vec{r}_V	Vectors in the vehicle-fixed and space-fixed coordinate system.
IA	Input axis of SAR or PIGA.
OA	Output axis of SAR or PIGA.
SA	Spin axis of SAR or PIGA.
SRA	Spin reference axis of SAR or PIGA.
LOS	Line of sight.

SYMBOL

DEFINITION

 α_n

Direction cosine angles

 β_n

(n=1, 2, 3).

 γ_n

.

 $\dot{\phi}_x$ Angular velocities of the vehicle
about the X_v , Y_v , and Z_v axes. $\dot{\phi}_y$ $\dot{\phi}_z$ θ_x

Euler angles.

 θ_y θ_z

.

 $\dot{\theta}_x$ Rates of change of the
Euler angles. $\dot{\theta}_y$ $\dot{\theta}_z$ θ_{ox} Redundant fourth angle in Euler's
four angle method. θ_{x1} The same as θ_x in Euler three angles except
that it is limited to a certain maximum
value in Euler's four angles.

.

 $\dot{\theta}_{ox}$

Rate of change of the redundant fourth angle.

 X_r Rotational coordinate system associated
with the Euler parameters. Y_r Z_r α Direction cosine angles defining the position
of X_r with respect to the X_v , Y_v , Z_v coordinate
system. β γ μ Angular rotation about the X_r axis. ξ

Euler parameters.

 η χ δ e_1

Cayley-Klein parameters

 e_2 e_3 e_4 S

Hamilton's quaternions.

 a b c

$C(i,j)$

A 3x3 orthogonal matrix relating space-fixed coordinates to vehicle-fixed coordinates.

\bar{i}_s

\bar{j}_s

\bar{k}_s

\bar{i}_v

\bar{k}_v

\bar{j}_v

Unit vectors in the space-fixed and the vehicle-fixed coordinate system.

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FOREWORD

This is a technical report prepared by the Advanced Studies Group, Electrical Engineering Department, Auburn University, toward fulfillment of Contract NAS8-20004 granted to Auburn Research Foundation, Auburn, Alabama. This contract was awarded January 19, 1965, and extended to January 18, 1966. It was further extended to April 18, 1966.

SUMMARY

The concept of the analytic platform inertial navigation system is reviewed and a summary of the construction and operation of an ideal Single Axis Reference and an ideal Pendulous Integrating Gyro Accelerometer (SAR and PIGA) is made in Chapters I and II. In Chapter III, four different methods are presented that relate the vehicle coordinates to those of the inertial reference and the mathematics, that enable one to calculate these transformations from the angular rates of the vehicle, are presented. Matrix operations and numerical integration techniques are presented in Appendices A and B as a review and as an aid in the calculation of the rotational transformations. The orthogonality and normality conditions are presented in Appendix C. Appendix D contains the definition of the orientation of the space-fixed and vehicle-fixed coordinate systems.

PERSONNEL

The following staff members of Auburn University have actively contributed in this study.

- C. L. Phillips - Professor of Electrical Engineering
Project Leader - January 19, 1965 to October 19, 1965
- J. L. Lowry - Professor of Electrical Engineering
Project Leader - October 19, 1965 to Present
- C. L. Connor - Graduate Assistant in Electrical Engineering
2/3 Time - September 1, 1965 to Present
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Full Time - January 19, 1965 to Present.
- R. J. Vinson - Graduate Assistant in Electrical Engineering
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I. INTRODUCTION*

D. W. Kelly

Inertial navigation** is concerned with the knowledge of where a vehicle is with respect to an inertially-fixed coordinate system. The only inputs to a classical inertial navigational device, other than initial conditions of position and velocity, are angular velocities and linear accelerations, which are measured by self-contained instruments on-board the vehicle. The definition of inertial navigation can be enlarged to include star-trackers, since the line of sight (LOS) to a star is an inertial reference line.

A. Stabilized Platform^{1,2,3,4}

The most common method of implementing an inertial navigation system utilizes a stabilized platform. The stable-platform or stable-table is a device which has the ability to keep a set of orthogonal measuring axes mechanically aligned with a given set of inertial axes, regardless of the angular movement of the vehicle. This alignment is accomplished by means of a gimballing system, torquers, servoloops and gyroscopes.

*All references are located at the end of the report.

**Inertial navigation is used on ballistic missiles, space boosters, tanks, aircraft, spacecraft, submarines, and is proposed for moon rover vehicles.

Figure I-1a shows a typical three gimbal platform layout with platform gimbals, gyroscopes, and accelerometers. Any angular motion of the innermost platform gimbal is sensed by the gyros. Sensors on the gyros provide a signal that is fed to the torque motors, which in turn maintain the space-fixed position of the inner gimbal. The accelerometers mounted on the inner gimbal sense the translational acceleration of the vehicle in the space-fixed coordinate system (the coordinate system of the inner gimbal).

The guidance computer uses the outputs of the accelerometers, along with the initial values of velocity and position, to calculate the instantaneous velocity and position of the vehicle relative to the inertially-fixed coordinate system. Figure I-2 shows a simplified block diagram of the system.

This is an excellent system inasmuch as it accomplishes its purpose to a high degree of accuracy, but it has disadvantages. By necessity the platform and gimbaling arrangement is spherical, which does not lend itself to economical packaging. It is rather heavy and requires a large power supply for the torque motors. It also has an operational limitation due to gimbal lock, a condition in which all of the gimbal planes are coplanar, leaving the inner gimbal with only two degrees of freedom. If, under this condition, a rotation is attempted about an axis perpendicular to this plane, the inner

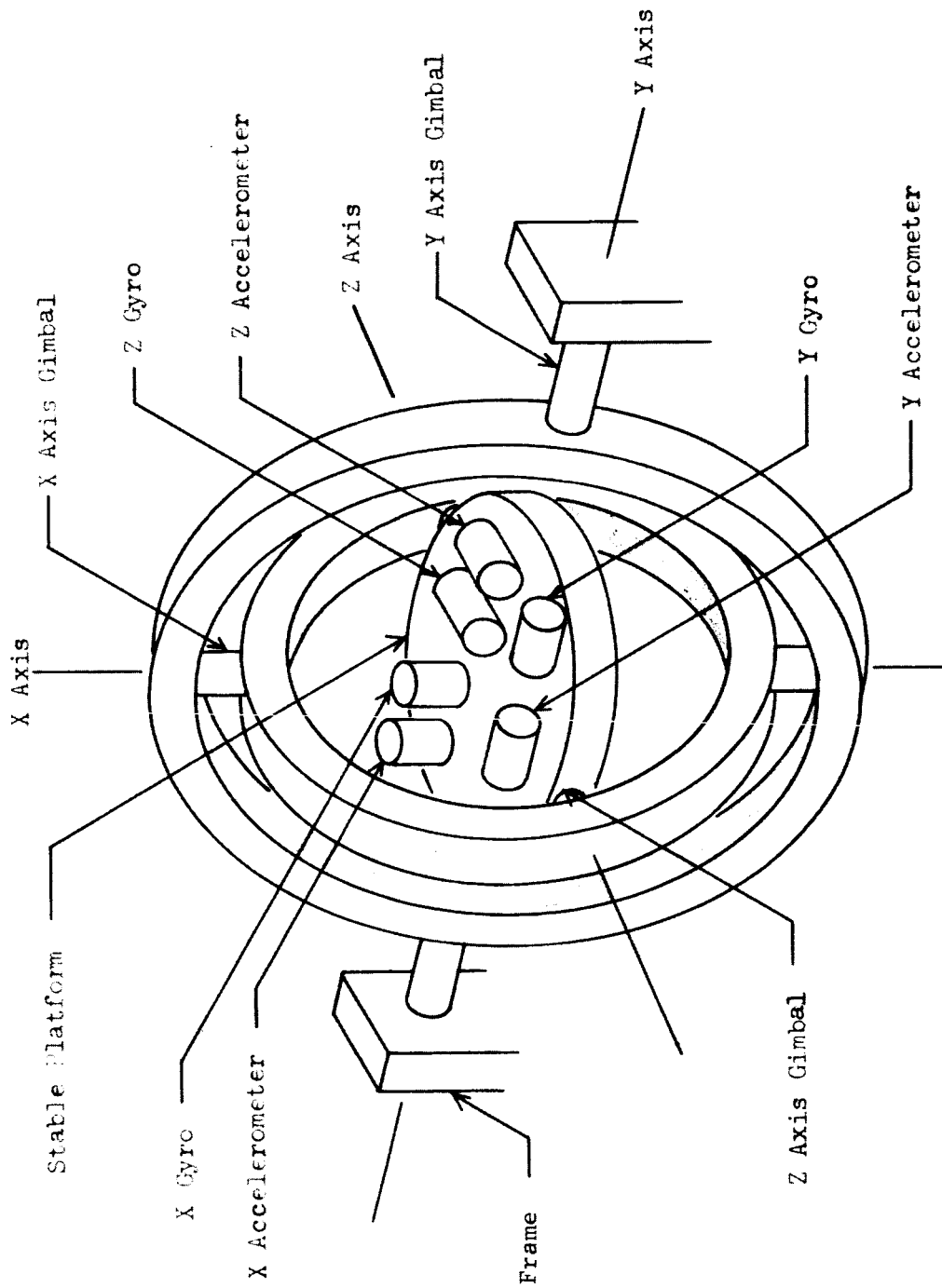


Fig. I-1a--A Three Gimbal Stable Platform Showing the Location of the Gyros and Accelerometers.

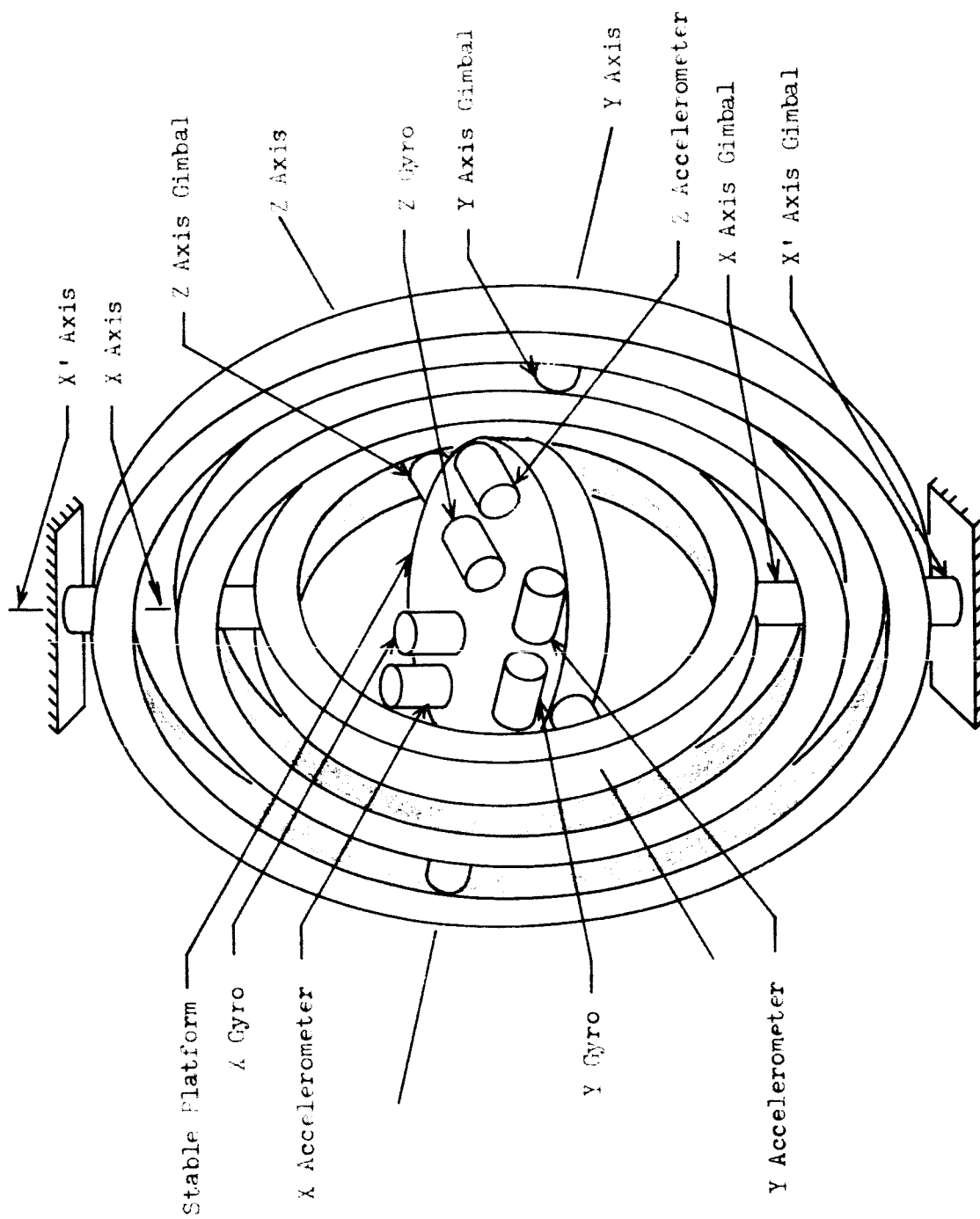


Fig. I-1b--A Four Gimbal Stable Platform Showing the Location of the Gyros and Accelerometers.

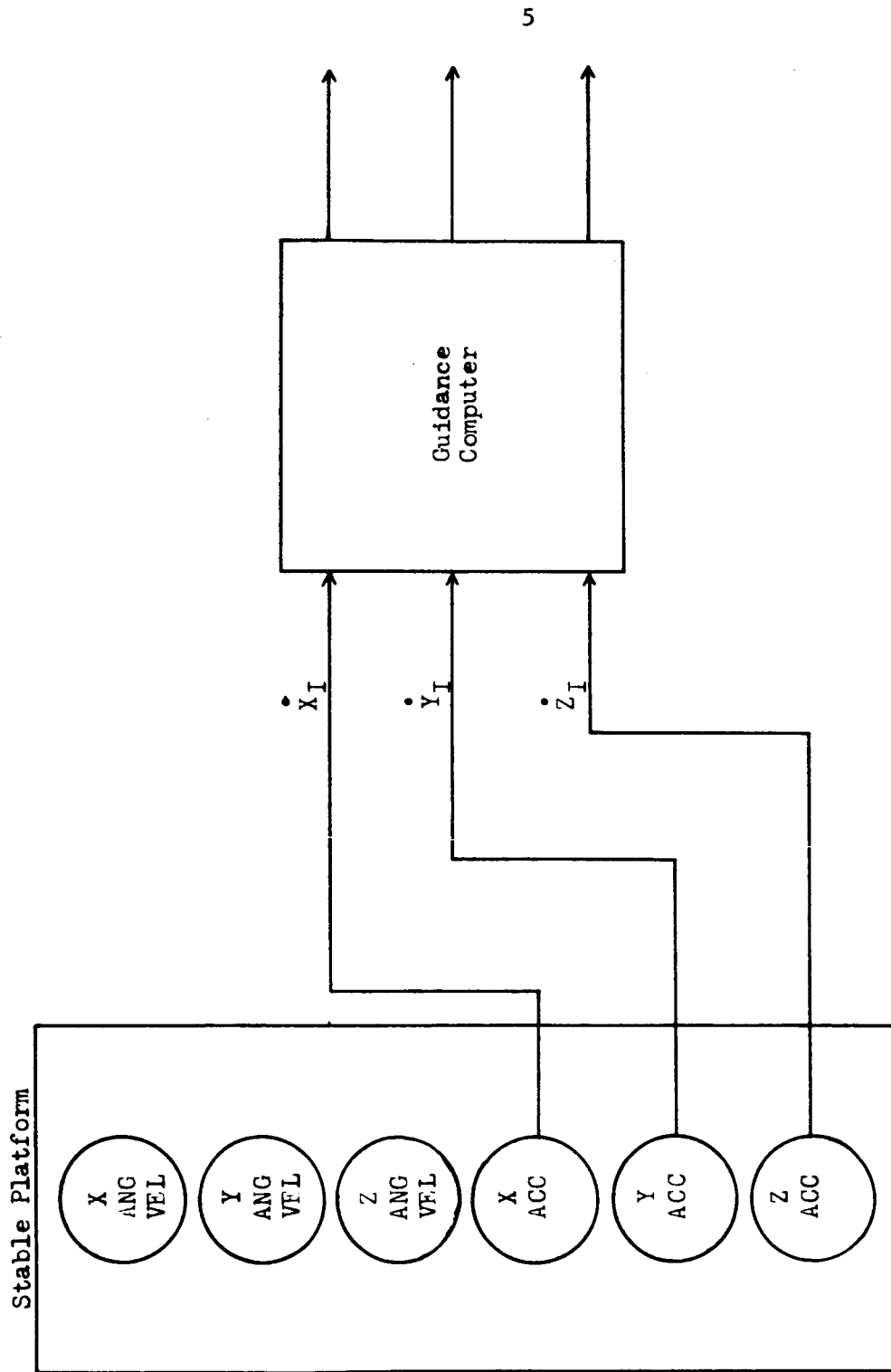


Fig. I-2--Simplified Block Diagram of the Stable Platform Inertial Navigation Method.

gimbal is forced to move about this perpendicular axis. Gimbal lock can be eliminated by the introduction of a redundant gimbal, and by limiting the movement of one gimbal so that the condition of coplanar gimbal planes cannot exist. Figure I-1b is a typical four-gimbal system.

Because of these disadvantages, it would be desirable to replace the present system with a system that would eliminate the platform, thereby removing the bulky spherical gimbaling arrangement and the large power supply. The new system must be capable of performing the same operations as the stable platform with the same degree of accuracy. The analytic platform discussed in the next section is a system proposed for this purpose.

B. Analytic Platform^{5,6,7,8,9}

The remainder of this report is a study of a system in which the mechanically stable platform is replaced by an analytic platform. The sensors are mounted directly to the vehicle and their outputs are fed into a coordinate transformation computer. Figure I-3 is a simplified block diagram of the analytic platform system. The outputs of three of the sensors are $\dot{\phi}_x$, $\dot{\phi}_y$, and $\dot{\phi}_z$, which are the angular rates of the vehicle with respect to the vehicle-fixed coordinate system. These outputs are used to generate a matrix that can in turn be used to transform the linear acceleration sensor outputs from the vehicle

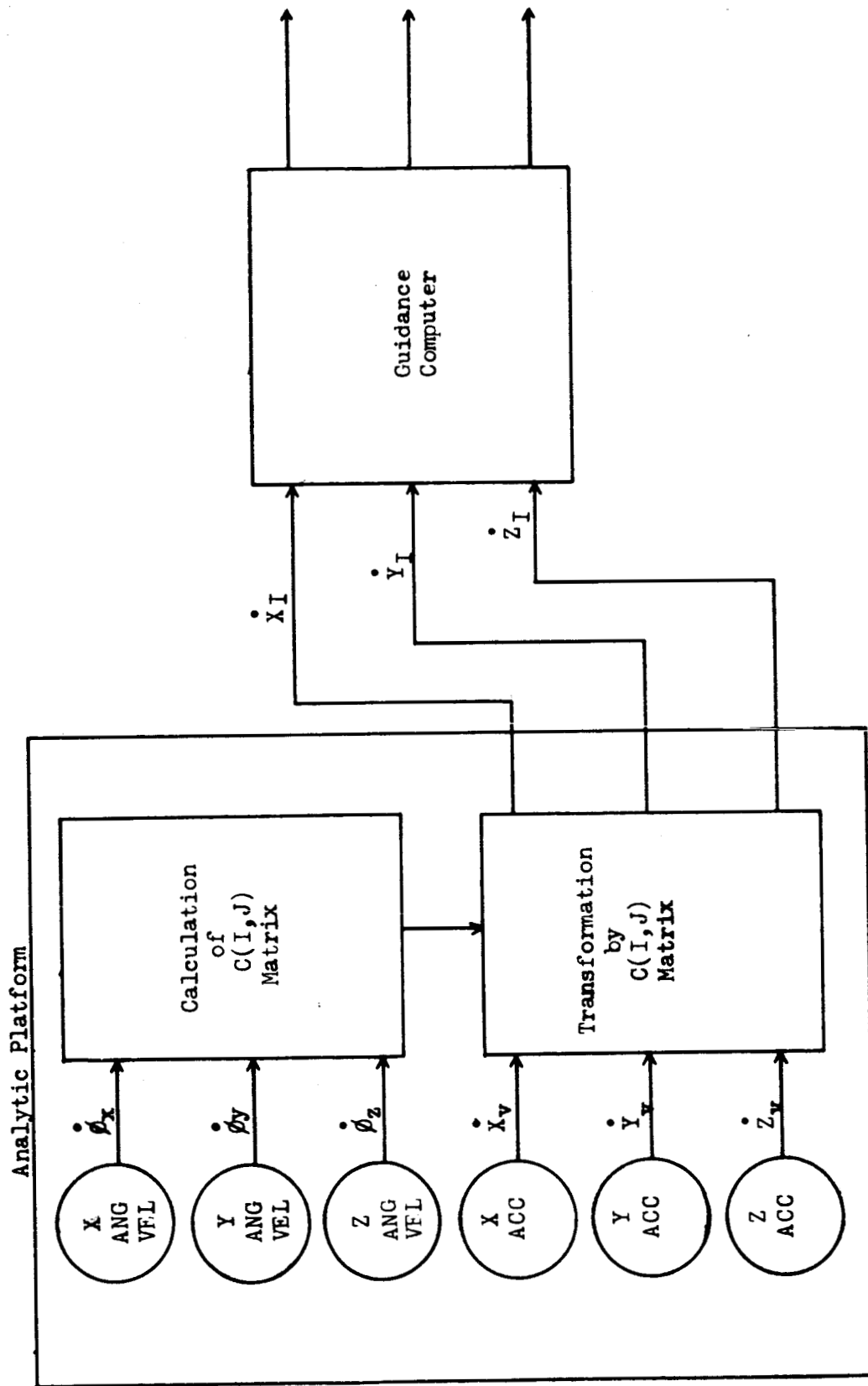


Fig. I-3--Simplified Block Diagram of the Analytic Platform Inertial Navigation Method.

coordinate system to the space-fixed coordinate system. The outputs from the coordinate transformation computer are the same as the outputs of the stable platform system, hence the name "analytic platform system". The remaining operations are the same as in the stable platform systems, as can be seen from a comparison of Figures I-2 and I-3.

Although the removal of the stable platform from the navigation system eliminates some disadvantages, it introduces others. This report is concerned with the problems of implementing the analytic platform.

II. GYROSCOPIC SENSORS

C. L. Connor, D. W. Kelly, and J. L. Lowry

A gyroscope may be defined broadly as a body rotating at a high angular velocity about an axis which is called its spin axis. The rotating body has the property of resisting any effort to change the direction of its spin axis, thus providing a reference from which either angular displacements or angular rates may be measured. The operation of a gyroscope is based on the following principles of gyro dynamics:

1. The gyro spin axis tends to remain fixed in space.
2. When a torque is applied to the gyro, a precession or angular rate results. This precession is a rotation about an axis orthogonal to both the gyro spin axis and the torque vector. The precession continues until the torque is removed or until the spin vector becomes aligned with the torque vector.

Precession of a rotating body is shown in Figure II-1.

There are many different types of gyroscopes which are usually classified according to construction. This chapter will discuss an ideal floated-type gyroscope known as a Single Axis Reference. It will also include a discussion of the special construction of a floated-type gyroscope for the purpose of measuring acceleration.

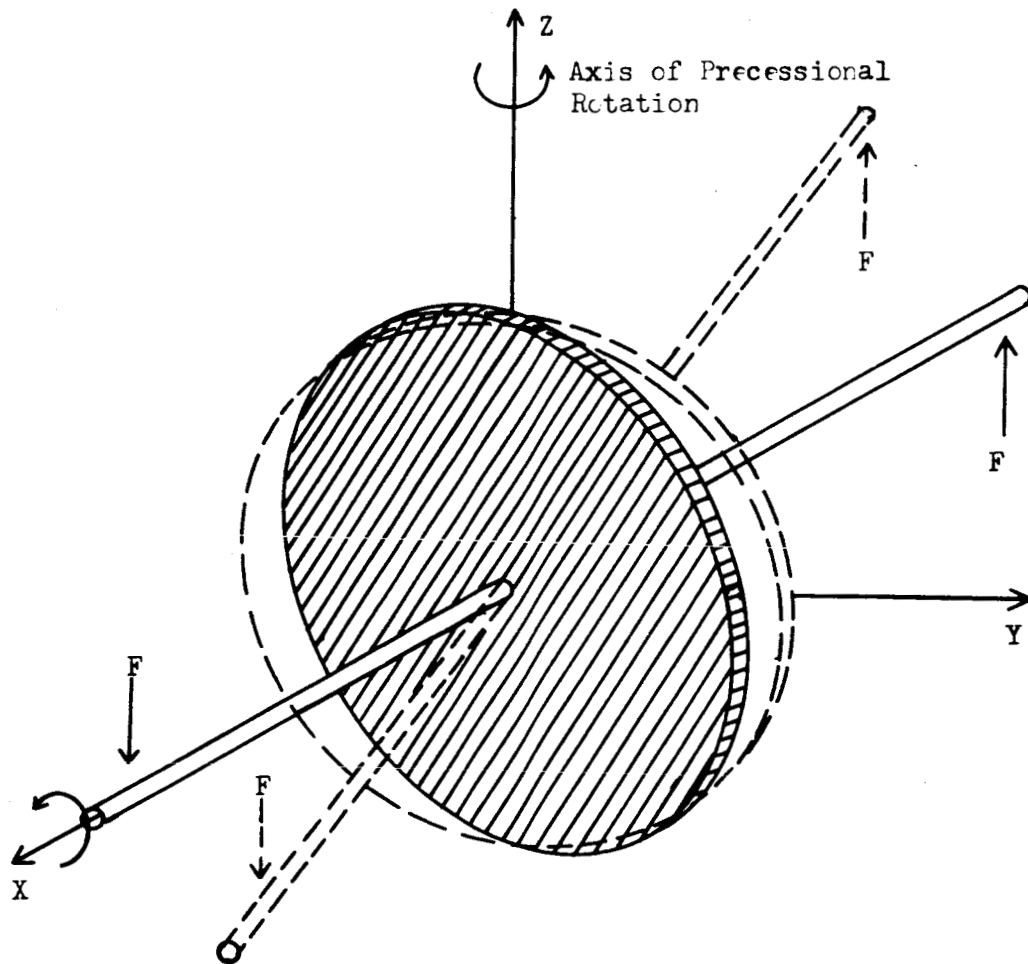


Fig. II-1--Precession of a Rotating Body.

A. Operation^{10,11}

The Single Axis Reference (SAR) is a device which measures angular displacement about a single reference axis through the center of the device. The SAR maintains this reference by means of the principles of gyro dynamics.

The SAR's construction is shown in Figure II-2 and is centered around the gyro wheel or rotor. The gyro wheel, due to its very nearly constant, high angular momentum, establishes the gyroscopic effects exhibited by the gyro. The gyro wheel is bearing mounted to, and is sealed inside of, the inner cylinder of the SAR. The inner cylinder of the SAR is suspended in a very nearly frictionless gas bearing inside the outer cylinder* of the SAR. The outer cylinder is mounted by trunion bearings to the housing, and is free to turn about the input axis IA (see Figure II-2). A null position pickoff device is mounted to the outer cylinder and is positioned so that it is sensitive to angular displacements of the inner cylinder with respect to a predetermined null position between the inner and outer cylinders. A torque motor (torquer) is mounted in the housing and can torque the outer cylinder about IA, producing a torque vector along IA. Also mounted along IA is an encoder, a device that measures the relative angular displacement between the housing and the outer cylinder. The encoder is described in the next section.

*The inner cylinder and outer cylinder are sometimes referred to as the float and case respectively.

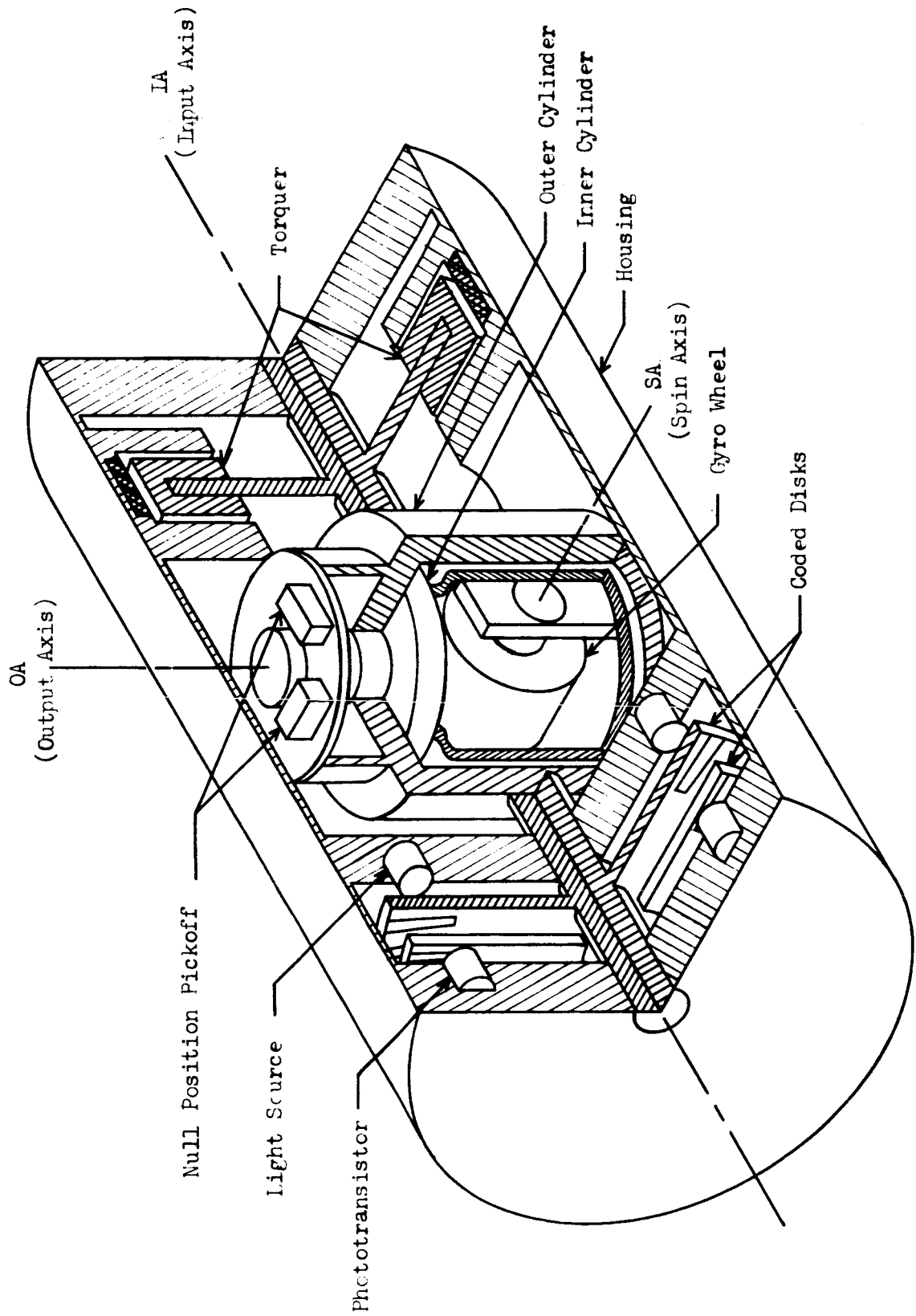


Fig. II-2--Simplified SAR

To understand the operation of an ideal SAR, consider its reaction to angular motion of the housing about the three axes, IA, OA and the spin reference axis. In this discussion it is assumed that the housing of the SAR is mounted to a heavier body whose motion will not be affected by the output of the torque motor of the SAR.

If there is an angular motion of the housing about IA, the outer cylinder will remain fixed with respect to the spin axis and the housing will move freely. If, due to friction or drag, the outer cylinder is torqued by the movement of the housing, then the gyro wheel and the inner cylinder will precess about OA. This precession will be sensed by the null position pickoff mounted at the end of the inner cylinder. The pickoff produces an electrical signal proportional to the angle sensed. This signal is amplified, compensated, and fed to the torquer which produces a torque on the outer cylinder about IA. This torque will oppose exactly the original torque due to friction, thus compensating for friction. The movement of the housing with respect to the outer cylinder will be sensed by the encoder and the encoder will give an output from the SAR.

If there is an angular motion of the housing about OA, the outer cylinder will move with the housing, while the inner cylinder will remain fixed. The angular displacement of the outer cylinder with respect to the inner cylinder will be sensed by the null position pickoff and an electrical signal will be generated. This signal is amplified, compensated and fed to the torquer which produces a torque on the outer cylinder about IA. This torque causes the gyro wheel and

the inner cylinder to precess about the OA, and to follow the angular motion of the housing. The alignment of the inner and outer cylinders with respect to the housing will remain the same, and the encoder will not give an output.

If there is an angular motion of the housing about the spin axis, both the outer cylinder and the inner cylinder will move with the housing. The alignments of the inner cylinder, the outer cylinder and the housing with respect to each other will remain the same, and the encoder will not give an output.

Translational motion or acceleration will not effect the output of the SAR, since it is designed with the center of mass of the gyro wheel, and the center of mass of the inner and outer cylinders, located at the intersection of IA, OA and the spin reference axis. The forces acting on the gyro wheel and outer cylinder due to translational acceleration will always produce zero torque, and, therefore, will not cause an angular motion of any part of the SAR.

The only motion of the SAR that will produce an output from the encoder is angular motion about IA. Therefore, the SAR is a single axis reference that measures angular displacement of the housing about its input axis, and its output is unaffected by any other motion, translational or rotational.

The Pendulous Integrating Gyro Accelerometer (PIGA) is a device which measures translational acceleration along an axis through the center of the device. The construction of a PIGA is the same as that of a SAR (see Figure II-2), except that it has an asymmetrical construction

that shifts the center of mass of the gyro wheel along its spin axis a specific distance from the intersection of IA, OA and the spin axis.

To understand the operation of an ideal PIGA, consider its reaction to translational acceleration along the three axes, IA, OA and the spin reference axis. Again, as in the discussion of the SAR, it is assumed that the housing of the PIGA is mounted to a heavier body whose motion will not be affected by the output of the torque motor of the PIGA.

If there is a translational acceleration along IA, the gyro wheel will experience a torque about OA due to the force of acceleration acting on the pendulous mass of the inner cylinder. This torque about OA will cause the outer cylinder to precess about IA at an angular rate that is proportional to the torque produced by the acceleration, and, therefore, proportional to the acceleration. Friction, or drag, between the housing and the outer cylinder is compensated for in the same manner as in the SAR. The output of the encoder is a measure of the angular displacement between the housing and the outer cylinder, and will be proportional to the integral of the acceleration.

If there is a translational acceleration along OA, the gyro wheel will experience a torque about IA. This torque will cause the inner cylinder to precess about IA, thus moving it away from its null position. The pickoff device will detect this movement and will produce a signal that is amplified, compensated and fed to the torque motor. The torque motor will produce a torque about IA that will balance out the torque due to acceleration in the OA direction. The final alignment of the inner cylinder, the outer cylinder and the housing with respect to each other will not be affected, and the encoder will not give an output.

If there is a translational acceleration along the spin axis, then the inner and outer cylinders will be accelerated without experiencing torques about OA or IA. The alignment of the inner cylinder, the outer cylinder and the housing with respect to each other will remain the same, and the encoder will not give an output.

The PIGA will react to rotational motion in exactly the same manner as did the SAR; that is, the PIGA will give an output for rotational motion about IA, but will not be affected by rotational motion about OA or about the spin reference axis. The output of the PIGA is a measure of the angular displacement between the housing and the outer cylinder. The rate of this angular displacement is proportional to the acceleration of the PIGA along IA. The angular displacement will, therefore, be proportional to the integral of the acceleration along IA of the PIGA plus the angular rotation of the housing of the PIGA about IA.

In order to use the PIGA as an accelerometer, the angle of rotation must be subtracted from its output. The PIGA will also measure the acceleration due to gravity, and, therefore, the output must be corrected for gravity.

When a PIGA is used on a stable platform, it will not experience rotational motion, and its output, after correction for gravity, will be the integral of the translation acceleration of the platform along the input axis of the PIGA. When the PIGA is mounted directly to the vehicle, its output must be corrected for angular rotation as well as for gravity. This correction will not present a problem since in the proposed

analytic platform, each PIGA will be used in conjunction with an SAR. The angular rotation, which will be measured by the SAR, gives the necessary information to correct the PIGA reading.

B. Quantization^{12,13}

The output of an SAR or of a PIGA can be obtained from an encoder that measures the angular displacement of the housing with respect to the outer cylinder. A typical input-output relation of an encoder is shown in Figure II-3. The output of the encoder is a pulse each time the angular position changes by a discrete amount (one pulse/one quantization level). The output pulses may be positive or negative corresponding to the direction of angle change. Due to the nature of the detectors used in an encoder, hysteresis, as shown in Figure II-3, exists in the detection of the angle of rotation. Typical quantization levels for SAR's are from 5 to 20 arcseconds.

Figure II-4 is a simplified drawing of a typical encoder. For a better understanding of the location and operation of the encoder in conjunction with the SAR, refer to Figure II-2. The coded discs are alternately transparent and opaque. One disc has $(2n)$ segments and the other has $(2n + 2)$ segments. One of the discs is mounted on the shaft of the outer cylinder and the other is attached to the housing. As the outer cylinder and housing rotate with respect to each other, the phototransistors detect the variation in the intensity of the light that passes through the discs. As one disc rotates $360/n$ degrees with respect to the other, the phototransistors will detect 360 degrees

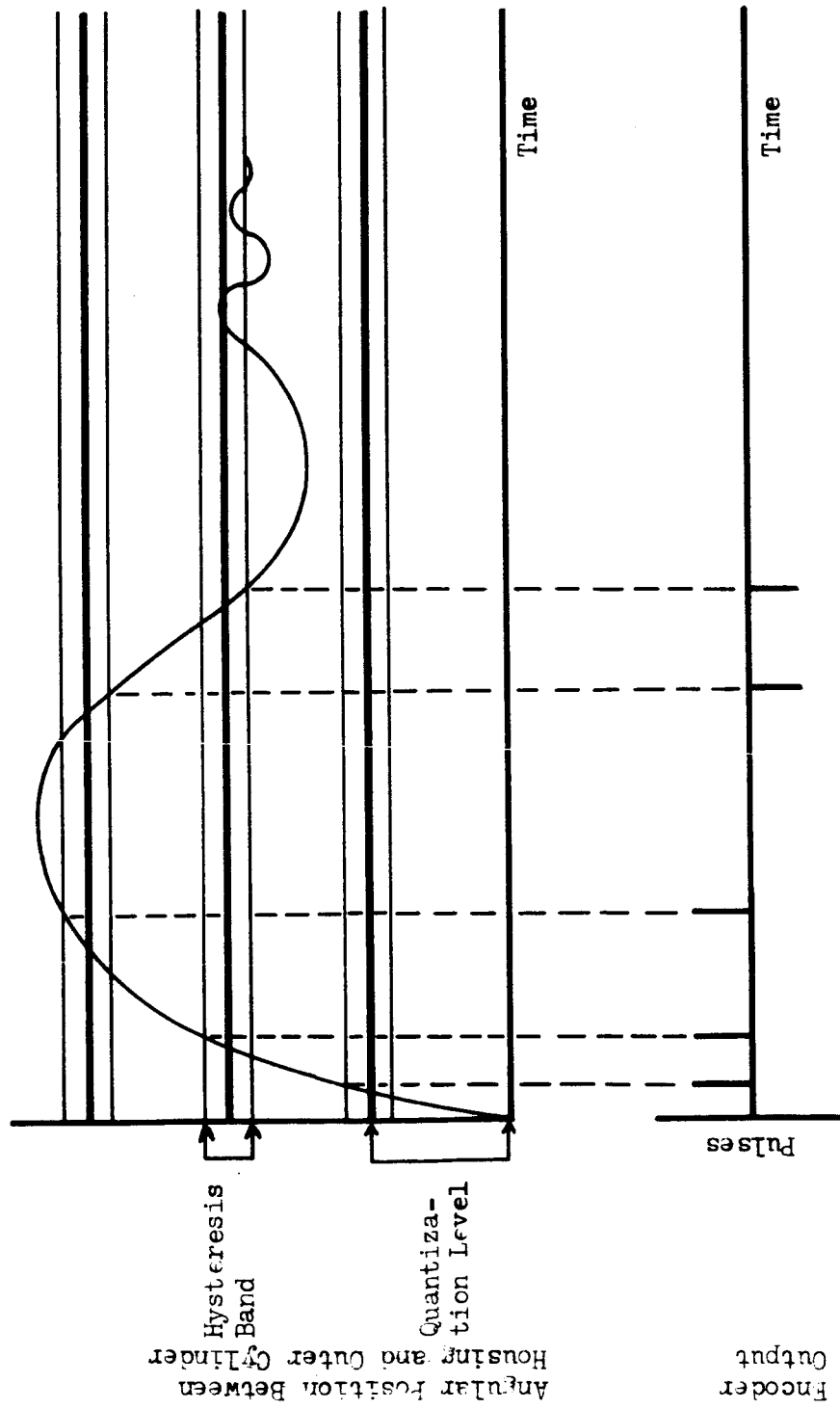


Fig. II-3--Simplified Drawing Showing Input-output Relations of a Typical Encoder.

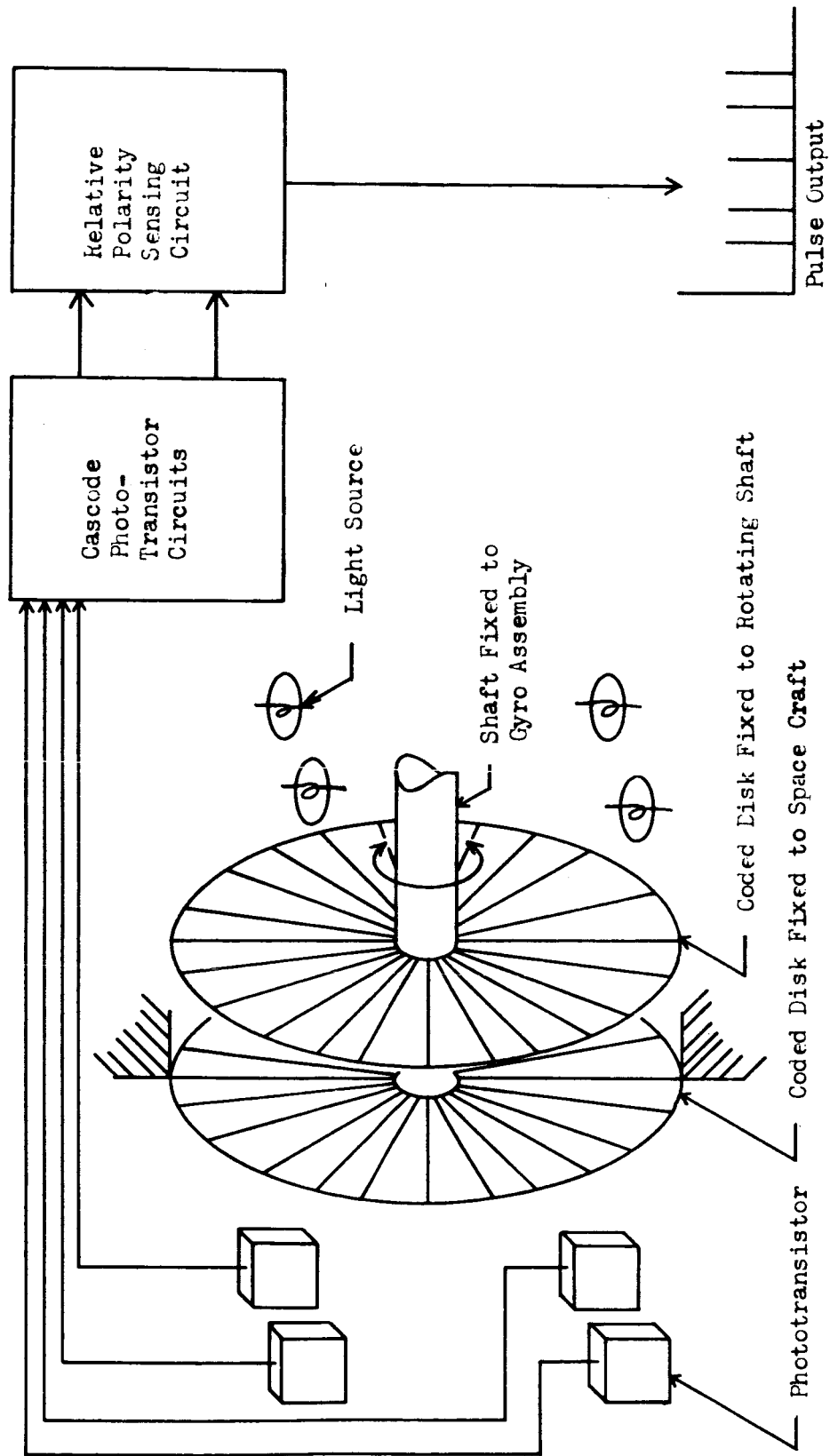


Fig. II-4--Simplified Drawing of the Encoder Showing the Coded Discs and Photo-device for Producing the Digital Output.

of rotation of the maximum light intensity point of the light that passes through the discs. The rotation of the maximum light intensity point is due to the vernier effect between the opaque segments of the two discs blocking the passage of light. The electronic circuits shown in Figure II-4 convert the outputs of the phototransistors to pulses that represent the angle of rotation.

III. TRANSFORMATION MATRIX

D. W. Kelly and R. J. Vinson

A. Introduction^{14,15,16}

The transformation matrix relating vehicle-fixed coordinates to space-fixed coordinates can be calculated by several different methods. These methods will be presented in this chapter along with the advantages and disadvantages of each.

The two coordinate systems will be designated by the subscripts v and s as shown in Figure III-1. The transformation matrix will be used to transform the velocity of the vehicle from the vehicle-fixed coordinate system to the space-fixed coordinate system. The only transformation necessary to perform this operation is a rotational transformation, and, therefore, the translation transformation relating the origins of the two coordinate system is not necessary. For this reason it is satisfactory to assume that the origin of the space-fixed coordinate system is translated to the center of mass of the vehicle and moves with the vehicle.

B. Rotational Matrix¹⁷

The rotational transformation matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \quad (\text{III-1})$$

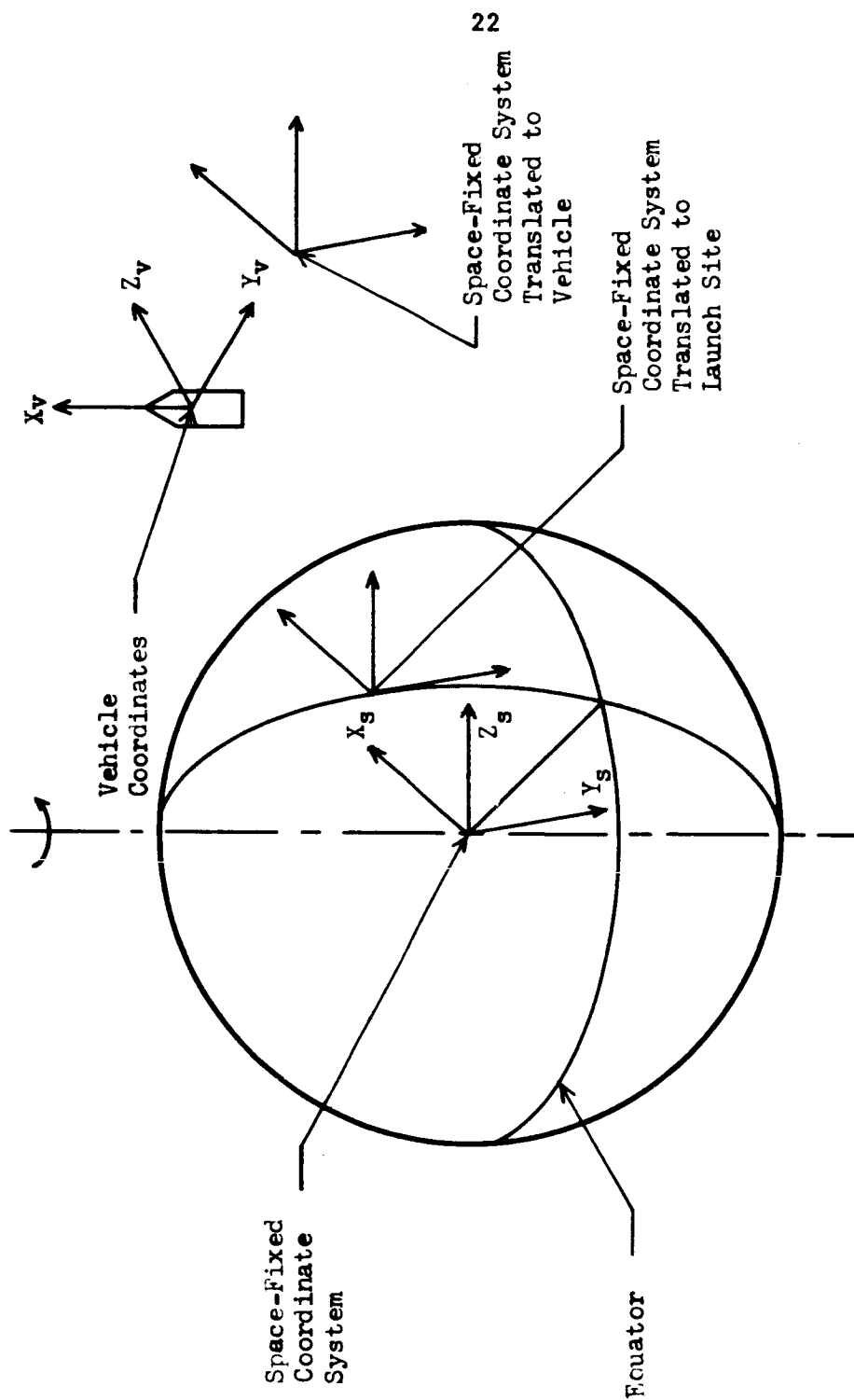


Fig. III-1--Vehicle-fixed and Space-fixed Coordinate Systems.

is a 3 x 3 matrix used to transform the velocity outputs of the PIGA's from the vehicle-fixed coordinate system to the space-fixed coordinate system, that is,

$$\begin{bmatrix} \dot{X}_s \\ \dot{Y}_s \\ \dot{Z}_s \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \dot{X}_v \\ \dot{Y}_v \\ \dot{Z}_v \end{bmatrix} \quad (\text{III-2})$$

This rotational matrix must be calculated in real time by means of a digital computer device on board the vehicle.

The rotational matrix may be calculated by any of the following basic concepts:

1. Direction cosines,
2. Euler angles (three and four gimbal concept),
3. Four parameter methods (Euler parameters, quaternions, and Cayley-Klein formulation).

In each case the rotational matrix is calculated using a set of differential equations which require as inputs the angular rates of rotation of the vehicle about the three vehicle-fixed coordinate axes, which can be obtained from the SAR's. The development of each as is pertinent to this study follows below. This presentation is intended only as a compilation of previous work.

1. DIRECTION COSINES

Consider a vector \bar{r} in the vehicle coordinate system as $\bar{r} = x_v \bar{i}_v + y_v \bar{j}_v + z_v \bar{k}_v$, where \bar{i}_v , \bar{j}_v , and \bar{k}_v are the unit vectors along the X_v, Y_v , and Z_v axes respectively. It is desired to express this same vector in the same space coordinate system as $\bar{r} = x_s \bar{i}_s + y_s \bar{j}_s + z_s \bar{k}_s$, where \bar{i}_s, \bar{j}_s , and \bar{k}_s are the unit vectors along the X_s, Y_s , and Z_s axes respectively. This can be accomplished by expressing x_s, y_s , and z_s in terms of x_v, y_v , and z_v as

$$\begin{aligned} x_s &= x_v \cos \alpha_1 + y_v \cos \alpha_2 + z_v \cos \alpha_3 \\ y_s &= x_v \cos \beta_1 + y_v \cos \beta_2 + z_v \cos \beta_3 \\ z_s &= x_v \cos \gamma_1 + y_v \cos \gamma_2 + z_v \cos \gamma_3 \end{aligned} \quad (\text{III-3})$$

where

$$\begin{aligned} \alpha_1 &= \text{angle between } \bar{i}_s \text{ and } \bar{i}_v \\ \alpha_2 &= \text{angle between } \bar{i}_s \text{ and } \bar{j}_v \\ \alpha_3 &= \text{angle between } \bar{i}_s \text{ and } \bar{k}_v \\ \beta_1 &= \text{angle between } \bar{j}_s \text{ and } \bar{i}_v \\ \beta_2 &= \text{angle between } \bar{j}_s \text{ and } \bar{j}_v \\ \beta_3 &= \text{angle between } \bar{j}_s \text{ and } \bar{k}_v \\ \gamma_1 &= \text{angle between } \bar{k}_s \text{ and } \bar{i}_v \\ \gamma_2 &= \text{angle between } \bar{k}_s \text{ and } \bar{j}_v \\ \gamma_3 &= \text{angle between } \bar{k}_s \text{ and } \bar{k}_v \end{aligned}$$

(III-1) can be expressed in matrix notation as

$$\begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} \quad (\text{III-4})$$

where the c_{ij} 's are the direction cosines and are

$$\begin{aligned} c_{1j} &= \cos \alpha_j \\ c_{2j} &= \cos \beta_j \\ c_{3j} &= \cos \gamma_j \quad \text{for } j = 1, 2, 3. \end{aligned} \quad (\text{III-5})$$

The matrix,

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \quad (\text{III-6})$$

is the rotational matrix that transforms the vector \bar{r} from the vehicle coordinate system to the space coordinate system. C is an orthogonal matrix, and, therefore, must satisfy all of the properties of an orthogonal matrix.

The elements of the transformation matrix can be expressed in terms of the dot product between the unit vectors of the two coordinate systems. The dot product is defined as: $\bar{a} \cdot \bar{b} = |\bar{a}| |\bar{b}| \cos \theta$, where θ is the angle between \bar{a} and \bar{b} . Thus, in terms of the dot product,

$$\begin{aligned}
c_{11} &= \bar{i}_v \cdot \bar{i}_s \\
c_{12} &= \bar{j}_v \cdot \bar{i}_s \\
c_{13} &= \bar{k}_v \cdot \bar{i}_s \\
c_{21} &= \bar{i}_v \cdot \bar{j}_s \\
c_{22} &= \bar{j}_v \cdot \bar{j}_s \\
c_{23} &= \bar{k}_v \cdot \bar{j}_s \\
c_{31} &= \bar{i}_v \cdot \bar{k}_s \\
c_{32} &= \bar{j}_v \cdot \bar{k}_s \\
c_{33} &= \bar{k}_v \cdot \bar{k}_s .
\end{aligned}
\tag{III-7}$$

The elements of the C matrix can be obtained by integrating if the time derivatives of the direction cosines are known. The time derivatives of the direction cosines can be related to the rates of rotation of the axes in the vehicle coordinate system, and to the instantaneous values of the direction cosines. Since the rates of rotation of the axes of the vehicle are measured, the on-board computer can calculate the instantaneous C matrix.

To show the relations between the time derivatives of the direction cosines, and the rates of rotation of the axes of the vehicle requires the definition,

$$\dot{\bar{\phi}} = \dot{\phi}_x \bar{i}_v + \dot{\phi}_y \bar{j}_v + \dot{\phi}_z \bar{k}_v ,
\tag{III-8}$$

where ϕ_x , ϕ_y , and ϕ_z are the angles of rotation about the axes of the vehicle and $\dot{\phi}_x$, $\dot{\phi}_y$, and $\dot{\phi}_z$ are the rates of rotation. The relations

between the unit vectors in the two coordinate systems are

$$\begin{bmatrix} \bar{i}_s \\ \bar{j}_s \\ \bar{k}_s \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \bar{i}_v \\ \bar{j}_v \\ \bar{k}_v \end{bmatrix} \quad (III-9)$$

Differentiating both sides of (III-8) with respect to time gives

$$\begin{bmatrix} \frac{d}{dt}(\bar{i}_s) \\ \frac{d}{dt}(\bar{j}_s) \\ \frac{d}{dt}(\bar{k}_s) \end{bmatrix} = \begin{bmatrix} \dot{c}_{11} & \dot{c}_{12} & \dot{c}_{13} \\ \dot{c}_{21} & \dot{c}_{22} & \dot{c}_{23} \\ \dot{c}_{31} & \dot{c}_{32} & \dot{c}_{33} \end{bmatrix} \begin{bmatrix} \bar{i}_v \\ \bar{j}_v \\ \bar{k}_v \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \frac{d}{dt}(\bar{i}_v) \\ \frac{d}{dt}(\bar{j}_v) \\ \frac{d}{dt}(\bar{k}_v) \end{bmatrix} \quad (III-10)$$

Since the length of a unit vector does not change, and since the direction of the unit vectors in the space coordinate system do not change; the time derivatives on the left of (III-10) are zero. Therefore,

$$\begin{bmatrix} \dot{c}_{11} & \dot{c}_{12} & \dot{c}_{13} \\ \dot{c}_{21} & \dot{c}_{22} & \dot{c}_{23} \\ \dot{c}_{31} & \dot{c}_{32} & \dot{c}_{33} \end{bmatrix} \begin{bmatrix} \bar{i}_v \\ \bar{j}_v \\ \bar{k}_v \end{bmatrix} = - \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \frac{d}{dt}(\bar{i}_v) \\ \frac{d}{dt}(\bar{j}_v) \\ \frac{d}{dt}(\bar{k}_v) \end{bmatrix} \quad (III-11)$$

From the definition of the derivative of a unit vector,

$$\frac{d}{dt}(\bar{i}_v) = \dot{\phi}_x \bar{i}_v = \begin{bmatrix} \bar{i}_v & \bar{j}_v & \bar{k}_v \\ \dot{\phi}_x & \dot{\phi}_y & \dot{\phi}_z \\ 1 & 0 & 0 \end{bmatrix} = \dot{\phi}_z \bar{j}_v - \dot{\phi}_y \bar{k}_v,$$

$$\frac{d}{dt}(\bar{j}_v) = \dot{\phi}_x \bar{j}_v = \begin{bmatrix} \bar{i}_v & \bar{j}_v & \bar{k}_v \\ \dot{\phi}_x & \dot{\phi}_y & \dot{\phi}_z \\ 0 & 1 & 0 \end{bmatrix} = \dot{\phi}_x \bar{k}_v - \dot{\phi}_z \bar{i}_v,$$

and

$$\frac{d}{dt}(\bar{k}_v) = \dot{\phi}_x \bar{k}_v = \begin{bmatrix} \bar{i}_v & \bar{j}_v & \bar{k}_v \\ \dot{\phi}_x & \dot{\phi}_y & \dot{\phi}_z \\ 0 & 0 & 1 \end{bmatrix} = \dot{\phi}_y \bar{i}_v - \dot{\phi}_x \bar{j}_v. \quad (\text{III-12})$$

Substituting these expressions in (III-11), the following set of equations may be obtained.

$$\dot{c}_{11} = c_{12} \dot{\phi}_z - c_{13} \dot{\phi}_y$$

$$\dot{c}_{12} = c_{13} \dot{\phi}_x - c_{11} \dot{\phi}_z$$

$$\dot{c}_{13} = c_{11} \dot{\phi}_y - c_{12} \dot{\phi}_x$$

$$\dot{c}_{21} = c_{22} \dot{\phi}_z - c_{23} \dot{\phi}_y$$

$$\dot{c}_{22} = c_{23} \dot{\phi}_x - c_{21} \dot{\phi}_z$$

$$\dot{c}_{23} = c_{21} \dot{\phi}_y - c_{22} \dot{\phi}_x$$

(III-13)

$$\begin{aligned}
 \dot{c}_{31} &= \dot{c}_{32}\dot{\phi}_z - \dot{c}_{33}\dot{\phi}_y \\
 \dot{c}_{32} &= \dot{c}_{33}\dot{\phi}_x - \dot{c}_{31}\dot{\phi}_z \\
 \dot{c}_{33} &= \dot{c}_{31}\dot{\phi}_y - \dot{c}_{32}\dot{\phi}_x
 \end{aligned}$$

These nine differential equations may be integrated by some numerical technique to obtain the new C rotational matrix.

2. EULER THREE ANGLES

Euler has shown that it is possible to go from one coordinate system to a second specified system by three distinct rotations. It is a simple matter to compute the matrix for each rotation, and then to multiply them together in the proper order to obtain the total rotational matrix.

Consider first a rotation about the Z_s axis by an angle ϕ_z . The result will be the $X'Y'Z'$ coordinate system as shown in Figure III-2a. Next a rotation about the X' axis by an angle θ_x as shown in Figure III-2b. The new coordinate system is $X''Y''Z''$. The last rotation is about the Y'' axis by an angle θ_y . Figure III-3c shown the final rotation and the coordinate system is the $X_vY_vZ_v$ system, which is the vehicle-fixed system.

Consider first, the rotation through the angle which is shown in Figure III-2a. If this is viewed from above, transformation of some arbitrary vector \bar{R} would appear as shown in Figure III-3a.

It can be seen from the geometry of Figure III-3a that the new X' , Y' , and Z' components are related to X_s , Y_s , and Z_s by the following matrix equation

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos \theta_z & \sin \theta_z & 0 \\ -\sin \theta_z & \cos \theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix} \quad . \quad (III-14)$$

Now the rotation of Figure III-2b may be viewed from the front along the X' axis, and Figure III-3b is obtained. From the geometry of this figure, it may be seen that

$$\begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_x & \sin \theta_x \\ 0 & -\sin \theta_x & \cos \theta_x \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} . \quad (\text{III-15})$$

The final rotation may be viewed from the front, along the Y'' axis, as in Figure III-3c. From the geometry of this figure it can be seen that

$$\begin{bmatrix} X_v \\ Y_v \\ Z_v \end{bmatrix} = \begin{bmatrix} \cos \theta_y & 0 & -\sin \theta_y \\ 0 & 1 & 0 \\ \sin \theta_y & 0 & \cos \theta_y \end{bmatrix} \begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix} . \quad (\text{III-16})$$

To determine the total transformation matrix which results from these three rotations, it is necessary only to multiply the three individual matrices in the correct order. If $(d\theta)$ is the product, then

$$\begin{bmatrix} X_v \\ Y_v \\ Z_v \end{bmatrix} = (d\theta) \begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix} . \quad (\text{III-17})$$

If (C) is the inverse of $(d\theta)$, then

$$\begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix} = (C) \begin{bmatrix} X_v \\ Y_v \\ Z_v \end{bmatrix} . \quad (\text{III-18})$$

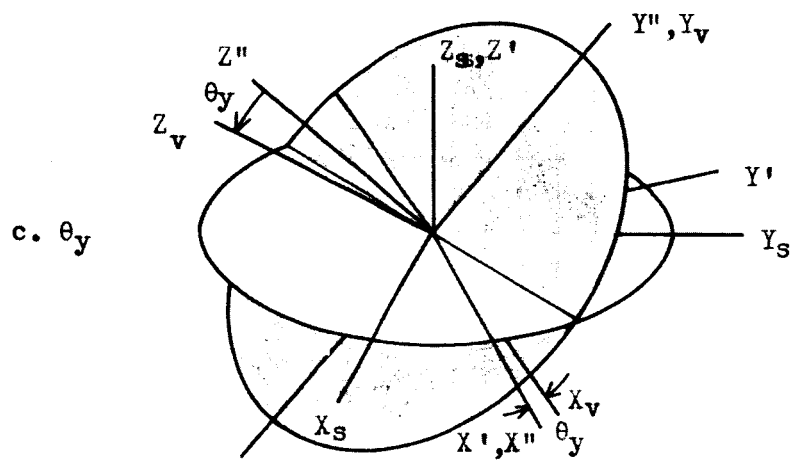
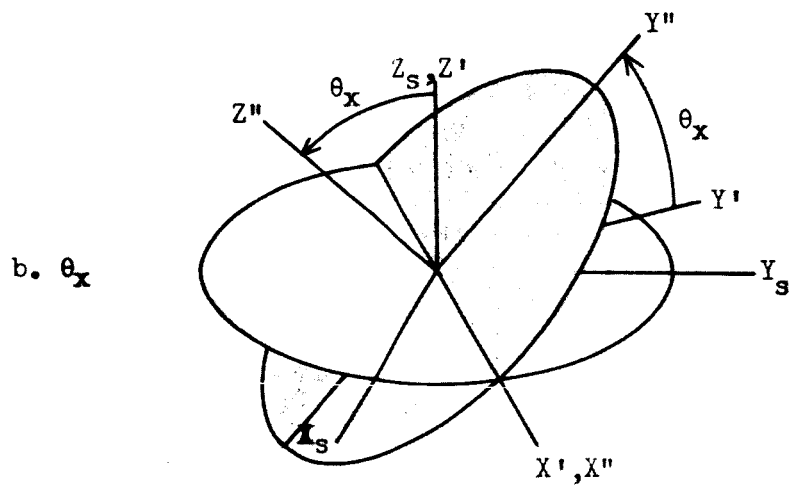
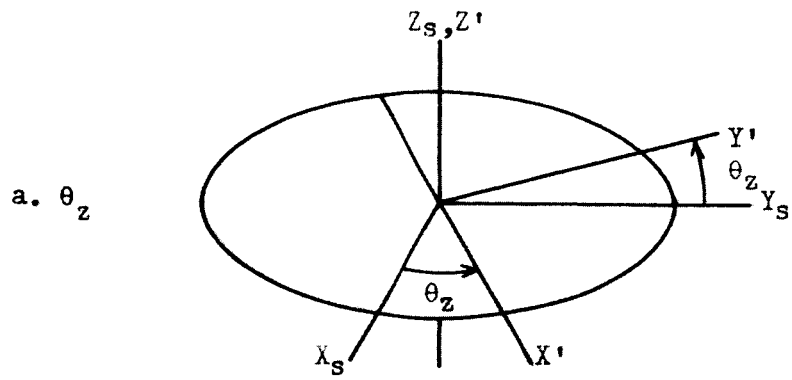


Fig. III-2--Fuler Angles.

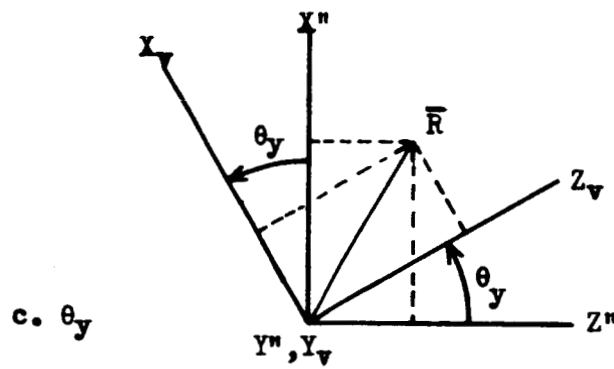
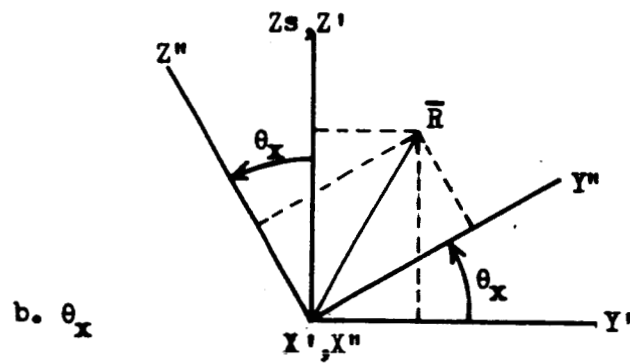
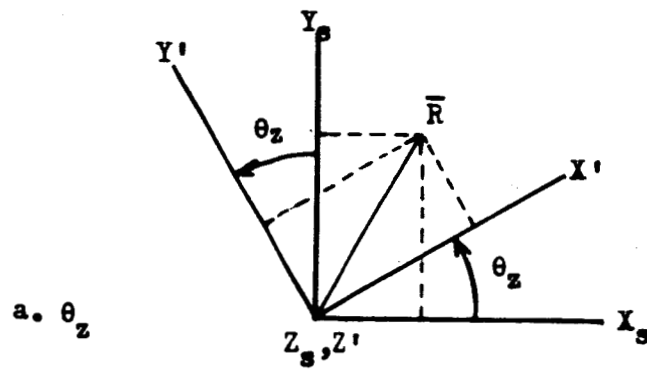


Fig. III-3--Transformation of Arbitrary Vector \bar{R}
Showing Euler Angles.

(C) may be determined in one of two ways; either obtain $(d\theta)$ and then take the transpose of $(d\theta)$, since $(d\theta)$ is orthogonal; or, take the transpose of the three matrices that give $(d\theta)$ and then multiply in reverse order. If the latter method is used, then

$$C = \begin{bmatrix} \cos \theta_z & -\sin \theta_z & 0 \\ \sin \theta_z & \cos \theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_x & -\sin \theta_x \\ 0 & \sin \theta_x & \cos \theta_x \end{bmatrix} \begin{bmatrix} \cos \theta_y & 0 & \sin \theta_y \\ 0 & 1 & 0 \\ -\sin \theta_y & 0 & \cos \theta_y \end{bmatrix}, \quad (\text{III-19})$$

which gives

$$C = \begin{bmatrix} \cos \theta_z \cos \theta_y & -\sin \theta_z \cos \theta_y & \cos \theta_z \sin \theta_y \\ -\sin \theta_z \sin \theta_x \sin \theta_y & & +\sin \theta_z \sin \theta_x \cos \theta_y \\ \sin \theta_z \cos \theta_y & \cos \theta_z \cos \theta_y & \sin \theta_z \sin \theta_y \\ +\cos \theta_z \sin \theta_x \sin \theta_y & & -\cos \theta_z \sin \theta_x \cos \theta_y \\ -\cos \theta_x \sin \theta_y & \sin \theta_y & \cos \theta_x \cos \theta_y \end{bmatrix}. \quad (\text{III-20})$$

Since the position of a coordinate system may be specified in terms of Euler angles, the rate of rotation of that coordinate system must be related to the rates of change of the Euler angles. This relationship will now be investigated.

It has been shown that a vector can be associated with a rate of rotation. The vector is directed along the instantaneous axis of rotation,

and is equal in magnitude to the rate of rotation. Thus, each of the Euler angle rates may be associated with a vector along the axis of rotation. Observe that the vector associated with the θ_z rotation of Figure III-3a is directed along the Z_I axis and points out of the page if θ_z is positive. Similarly, the rate of rotation due to the θ_x vector is in the positive X' direction. Finally, a rate of rotation due to the θ_y vector is in the positive Y'' direction. These Euler angle rates must be transformed to the vehicle coordinate system in order to relate the vehicle angular rates to the Euler angle rates. The rotation matrices (III-19) previously derived may be used. Therefore, the $\dot{\theta}_z$ Euler angle rate gives

$$\begin{bmatrix} \dot{\phi}_{xv} \\ \dot{\phi}_{yv} \\ \dot{\phi}_{zv} \end{bmatrix} = \begin{bmatrix} \cos \theta_y & 0 & -\sin \theta_y \\ 0 & 1 & 0 \\ \sin \theta_y & 0 & \cos \theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_x & \sin \theta_x \\ 0 & -\sin \theta_x & \cos \theta_x \end{bmatrix} \begin{bmatrix} \cos \theta_z & \sin \theta_z & 0 \\ -\sin \theta_z & \cos \theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{bmatrix} \quad (\text{III-21})$$

or

$$\begin{aligned} \dot{\phi}_{xv} &= (-\cos \theta_x \sin \theta_y) \dot{\theta}_z, \\ \dot{\phi}_{yv} &= (\sin \theta_x) \dot{\theta}_z, \\ \dot{\phi}_{zv} &= (\cos \theta_x \cos \theta_y) \dot{\theta}_z. \end{aligned} \quad (\text{III-22})$$

The $\dot{\theta}_x$ Euler angle rate gives

$$\begin{bmatrix} \dot{\phi}_{xv} \\ \dot{\phi}_{yv} \\ \dot{\phi}_{zv} \end{bmatrix} = \begin{bmatrix} \cos \theta_y & 0 & -\sin \theta_y \\ 0 & 1 & 0 \\ \sin \theta_y & 0 & \cos \theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_x & \sin \theta_x \\ 0 & -\sin \theta_x & \cos \theta_x \end{bmatrix} \begin{bmatrix} \dot{\theta}_x \\ 0 \\ 0 \end{bmatrix}, \quad (\text{III-23})$$

or

$$\begin{aligned}\dot{\phi}_{xv} &= (\cos \theta_y) \dot{\theta}_x, \\ \dot{\phi}_{yv} &= 0, \\ \dot{\phi}_{zv} &= (\sin \theta_y) \dot{\theta}_x.\end{aligned}\tag{III-24}$$

The $\dot{\theta}_y$ Euler angle rate gives

$$\begin{bmatrix} \dot{\phi}_{xv} \\ \dot{\phi}_{yv} \\ \dot{\phi}_{zv} \end{bmatrix} = \begin{bmatrix} \cos \theta_y & 0 & -\sin \theta_y \\ 0 & 1 & 0 \\ \sin \theta_y & 0 & \cos \theta_y \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_y \\ 0 \end{bmatrix},\tag{III-25}$$

or

$$\begin{aligned}\dot{\phi}_{xv} &= 0, \\ \dot{\phi}_{yv} &= \dot{\theta}_y, \\ \dot{\phi}_{zv} &= 0.\end{aligned}\tag{III-26}$$

Adding the components of the vehicle angular rates, the following matrix equation is derived.

$$\begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} = \begin{bmatrix} \cos \theta_y & 0 & -\sin \theta_y \cos \theta_x \\ 0 & 1 & \sin \theta_x \\ \sin \theta_y & 0 & \cos \theta_y \cos \theta_x \end{bmatrix} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix}.\tag{III-27}$$

But the vehicle angular rates are known, and the Euler angle rates are needed. They can be determined by taking the inverse of the above

matrix. Therefore,

$$\begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} = \begin{bmatrix} \cos \theta_y & 0 & \sin \theta_y \\ \sin \theta_y \tan \theta_x & 1 & -\cos \theta_y \tan \theta_y \\ -\sin \theta_y / \cos \theta_x & 0 & \cos \theta_y / \cos \theta_x \end{bmatrix} \begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} \quad (III-28)$$

From these equations, it is easy to see the difficulties which arise when θ_x approaches 90° . For this value of θ_x , both $\dot{\theta}_y$ and $\dot{\theta}_z$ are indeterminate. The three differential equations obtained from (III-28) may be integrated to obtain the new Euler angles. These may be substituted into (III-20) to obtain the new C rotational matrix.

3. EULER FOUR ANGLES

In an effort to avoid the singular point as described in Euler three angles, a redundant fourth angle will be added and the rotation of θ_x will be limited. This fourth angle, θ_{ox} , will be a rotation about the x_v axis and will have an unlimited rotation. The previous θ_x will be called θ_{x1} . The C matrix for the four angle system will be

$$C = \begin{bmatrix} \text{C Matrix} \\ \text{For Three} \\ \text{Euler Angle Case} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{ox} & \sin \theta_{ox} \\ 0 & -\sin \theta_{ox} & \cos \theta_{ox} \end{bmatrix} \quad (III-29)$$

Therefore the new C matrix will be

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \quad (III-30)$$

where

$$c_{11} = \cos \theta_z \cos \theta_y - \sin \theta_z \sin \theta_{x1} \sin \theta_y,$$

$$c_{12} = -\sin \theta_z \cos \theta_{x1} \cos \theta_{ox} + \cos \theta_z \sin \theta_y \sin \theta_{ox} \\ + \sin \theta_z \sin \theta_{x1} \cos \theta_y \sin \theta_{ox},$$

$$c_{13} = \sin \theta_z \cos \theta_{x1} \cos \theta_{ox} + \cos \theta_z \sin \theta_y \cos \theta_{ox} \\ + \sin \theta_z \sin \theta_{x1} \cos \theta_y \cos \theta_{ox},$$

$$c_{21} = \sin \theta_z \cos \theta_y + \cos \theta_z \sin \theta_{x1} \sin \theta_y,$$

$$c_{22} = \cos \theta_z \cos \theta_{x1} \cos \theta_{ox} + \sin \theta_z \sin \theta_y \sin \theta_{ox} \\ - \cos \theta_z \sin \theta_{x1} \cos \theta_y \sin \theta_{ox},$$

$$c_{23} = -\cos \theta_z \cos \theta_{x1} \sin \theta_{ox} + \sin \theta_z \sin \theta_y \cos \theta_{ox} \\ - \cos \theta_z \sin \theta_{x1} \cos \theta_y \cos \theta_{ox},$$

$$c_{31} = -\cos \theta_{x1} \sin \theta_y,$$

$$c_{32} = \sin \theta_{x1} \cos \theta_{ox} + \cos \theta_{x1} \cos \theta_y \sin \theta_{ox},$$

$$c_{33} = -\sin \theta_{x1} \sin \theta_{ox} + \cos \theta_{x1} \cos \theta_y \cos \theta_{ox}.$$

Now each of the Euler angle rates must be related to the vehicle angular rates. As was shown in the preceeding section on Euler three angles, each of the Euler angles may be related to the vehicle angular rates by transformation matrices. Therefore, the components of the vehicle angular rates caused by $\dot{\theta}_z$ are given by:

$$\begin{bmatrix} \dot{\phi}_{xv} \\ \dot{\phi}_{yv} \\ \dot{\phi}_{zv} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{ox} & \sin \theta_{ox} \\ 0 & -\sin \theta_{ox} & \cos \theta_{ox} \end{bmatrix} \begin{bmatrix} \cos \theta_y & 0 & -\sin \theta_y \\ 0 & 1 & 0 \\ \sin \theta_y & 0 & \cos \theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{x1} & \sin \theta_{x1} \\ 0 & -\sin \theta_{x1} & \cos \theta_{x1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{bmatrix} \quad (\text{III-31})$$

which reduces to:

$$\begin{aligned} \dot{\phi}_{xv} &= (-\sin \theta_y \cos \theta_{x1}) \dot{\theta}_z, \\ \dot{\phi}_{yv} &= (\sin \theta_{x1} \cos \theta_{ox} + \cos \theta_y \cos \theta_{x1} \sin \theta_{ox}) \dot{\theta}_z, \\ \dot{\phi}_{zv} &= (-\sin \theta_{x1} \sin \theta_{ox} + \cos \theta_y \cos \theta_{x1} \cos \theta_{ox}) \dot{\theta}_z. \end{aligned} \quad (\text{III-32})$$

Similarly, the components of the vehicle angular rates caused by $\dot{\theta}_{x1}$ are given by:

$$\begin{bmatrix} \dot{\phi}_{xv} \\ \dot{\phi}_{yv} \\ \dot{\phi}_{zv} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{ox} & \sin \theta_{ox} \\ 0 & -\sin \theta_{ox} & \cos \theta_{ox} \end{bmatrix} \begin{bmatrix} \cos \theta_y & 0 & -\sin \theta_y \\ 0 & 1 & 0 \\ \sin \theta_y & 0 & \cos \theta_y \end{bmatrix} \begin{bmatrix} \dot{\theta}_{x1} \\ 0 \\ 0 \end{bmatrix} \quad (\text{III-33})$$

which reduces to:

$$\begin{aligned} \dot{\phi}_{xv} &= (\cos \theta_y) \dot{\theta}_{x1}, \\ \dot{\phi}_{yv} &= (\sin \theta_{ox} \sin \theta_y) \dot{\theta}_{x1}, \\ \dot{\phi}_{zv} &= (\cos \theta_{ox} \sin \theta_y) \dot{\theta}_{x1}. \end{aligned} \quad (\text{III-34})$$

The components of the vehicle angular rates caused by θ_y are given by

$$\begin{bmatrix} \dot{\phi}_{xv} \\ \dot{\phi}_{yv} \\ \dot{\phi}_{zv} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{ox} & \sin \theta_{ox} \\ 0 & -\sin \theta_{ox} & \cos \theta_{ox} \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\theta}_y \cos \theta_{ox} \\ -\dot{\theta}_y \sin \theta_{ox} \end{bmatrix} \quad (\text{III-35})$$

The components of the vehicle angular rates caused by θ_{ox} are given by:

$$\begin{aligned} \dot{\phi}_{xv} &= \dot{\theta}_{ox}, \\ \dot{\phi}_{yv} &= 0, \\ \dot{\phi}_{zv} &= 0. \end{aligned} \quad (\text{III-36})$$

Adding the components of the vehicle angular rates caused by all four of the Euler angle rates gives:

$$\begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cos \theta_y & -\sin \theta_y \cos \theta_{x1} \\ 0 & \cos \theta_{ox} & \sin \theta_{ox} \sin \theta_y & (\sin \theta_{x1} \cos \theta_{ox} + \cos \theta_y \cos \theta_{x1} \sin \theta_{ox}) \\ 0 & -\sin \theta_{ox} & \cos \theta_{ox} \sin \theta_y & (\cos \theta_y \cos \theta_{x1} \cos \theta_{ox} - \sin \theta_{x1} \sin \theta_{ox}) \end{bmatrix} \begin{bmatrix} \dot{\theta}_{ox} \\ \dot{\theta}_y \\ \dot{\theta}_{x1} \\ \dot{\theta}_z \end{bmatrix} \quad (\text{III-37})$$

which may be rewritten as:

$$\begin{bmatrix} \dot{\phi}_x - \dot{\theta}_{ox} \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} = \begin{bmatrix} \cos \theta_y & 0 & -\sin \theta_y \cos \theta_{x1} \\ \sin \theta_{ox} \sin \theta_y & \cos \theta_{ox} & (\sin \theta_{x1} \cos \theta_{ox} + \cos \theta_y \cos \theta_{x1} \cos \theta_{ox}) \\ \cos \theta_{ox} \sin \theta_y & -\sin \theta_{ox} & (\cos \theta_y \cos \theta_{x1} \cos \theta_{ox} - \sin \theta_{x1} \sin \theta_{ox}) \end{bmatrix} \begin{bmatrix} \dot{\theta}_{x1} \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \quad (\text{III-38})$$

To determine the rate of change of the Euler angles with respect to the vehicle angular rates, the inverse of the above matrix must be determined.

Therefore,

$$\begin{bmatrix} \dot{\theta}_{x1} \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \dot{\phi}_x - \dot{\theta}_{ox} \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix}, \quad (\text{III-39})$$

where

$$\begin{aligned} b_{11} &= \cos \theta_y, \\ b_{12} &= \sin \theta_y \sin \theta_{ox}, \\ b_{13} &= \sin \theta_y \cos \theta_{ox}, \\ b_{21} &= \sin \theta_y \tan \theta_{x1}, \\ b_{22} &= \cos \theta_y \tan \theta_{x1} \sin \theta_{ox} + \cos \theta_{ox}, \\ b_{23} &= -\cos \theta_y \tan \theta_{x1} \cos \theta_{ox} - \sin \theta_{ox}, \\ b_{31} &= -\sin \theta_y / \cos \theta_{x1}, \\ b_{32} &= \cos \theta_y \sin \theta_{ox} / \cos \theta_{x1}, \\ b_{33} &= \cos \theta_y \cos \theta_{ox} / \cos \theta_{x1}. \end{aligned}$$

Three differential equations may be obtained from (III-39), but it should be noted that besides the angular velocities of the vehicle, an additional angular rate of $\dot{\theta}_{ox}$ must be known. This last rate may be obtained from the gimbaling arrangement between the engines and vehicle.

4. Four Parameters

In this section, three different methods of obtaining the four parameters are developed. These three methods (Euler parameters, Cayley-Klein parameter, and quaternions) lead to the same set of differential equations. The last part of this section develops the relationship between the four parameters and angular velocities.

Euler Parameters

Euler's Theorem: Any real rotation may be expressed as a rotation through some angle, about some fixed axis.¹⁸

Consider the angles μ , α , β , and γ where μ is the angle of rotation and α , β , and γ specify the fixed axis of rotation. Let $X_R Y_R Z_R$ be an additional coordinate system fixed at the origin of the $X_V Y_V Z_V$ system. The X_R axis is the axis of rotation and makes the angles of α , β , and γ with X_V , Y_V , and Z_V axes respectively. In addition the Y_R axis is restricted to the $X_V Y_V$ plane and, therefore, is perpendicular to the Z_V axis. The rotation of the $X_V Y_V Z_V$ system through the angle μ may be viewed as the result of three rotations: (1) rotation of the $X_V Y_V Z_V$ system into coincidence with the $X_R Y_R Z_R$ system (A rotation), (2) rotation through the angle μ about the X_R axis (R rotation), and (3) the reverse of (1) to restore the original separation of the $X_V Y_V Z_V$ system and the $X_R Y_R Z_R$ system (A^I rotation). The matrix for each of these transformations is developed, and the three may be multiplied together to express the total transformation.

Consider the first transformation from the $X_V Y_V Z_V$ system into the $X_R Y_R Z_R$ system, where α , β , and γ are the angles between the X_R axis and the X_V , Y_V , and Z_V axes respectively, then the a_{11} , a_{12} , and a_{13} are direction cosines, and since the Y_R axis is perpendicular to the Z_V axis, $a_{23} = 0$. Therefore, part of the first transformations is

$$A = \begin{bmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \quad (\text{III-40})$$

It is possible from orthogonality conditions to complete A.

$$A = \begin{bmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \pm \cos \beta \csc \gamma & \pm \cos \alpha \csc \gamma & 0 \\ \pm \cos \alpha \cot \gamma & \pm \cos \beta \cot \gamma & \pm \sin \gamma \end{bmatrix}. \quad (\text{III-41})$$

Since the above matrix must reduce to the identity matrix when α becomes zero and γ and β are equal to 90° , the correct sign may be determined. The result is

$$A = \begin{bmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ -\cos \beta \csc \gamma & \cos \alpha \csc \gamma & 0 \\ -\cos \alpha \cot \gamma & -\cos \beta \cot \gamma & \sin \gamma \end{bmatrix}. \quad (\text{III-42})$$

The second rotation, through the angle μ , about the X_r axis is simply

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \mu & \sin \mu \\ 0 & -\sin \mu & \cos \mu \end{bmatrix} . \quad (\text{III-43})$$

The last rotation is the inverse of (A) or $(A)^I$. Thus, the general transformation is given by

$$C = (A)^I(R)(A) \quad (\text{III-44})$$

This is similarity transformation and the trace of a matrix is invariant under a similarity transformation, that is,

$$c_{11} + c_{22} + c_{33} = 1 + 2 \cos \mu \quad (\text{III-45})$$

so the angle of rotation may be obtained directly from the diagonal elements. Carrying out the operations in (III-44) gives

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} , \quad (\text{III-46})$$

where

$$\begin{aligned}
 c_{11} &= 1 - 2 \sin^2 (\mu/2) \sin^2 \alpha \\
 c_{12} &= 2(\sin^2 (\mu/2) \cos \alpha \cos \beta + \sin (\mu/2) \cos (\mu/2) \cos \gamma) \\
 c_{13} &= 2(\cos \alpha \cos \gamma \sin^2 (\mu/2) - \sin (\mu/2) \cos (\mu/2) \cos (\mu/2) \cos \beta) \\
 c_{21} &= 2(\sin^2 (\mu/2) \cos \alpha \cos \beta - \sin (\mu/2) \cos (\mu/2) \cos \gamma) \\
 c_{22} &= 1 - 2 \sin^2 (\mu/2) \sin^2 \beta \\
 c_{23} &= 2(\sin^2 (\mu/2) \cos \beta \cos \gamma + \sin (\mu/2) \cos (\mu/2) \cos \alpha) \\
 c_{31} &= 2(\cos \alpha \cos \gamma \sin^2 (\mu/2) + \sin (\mu/2) \cos (\mu/2) \cos \beta) \\
 c_{32} &= 2(\sin^2 (\mu/2) \cos \beta \cos \gamma - \sin (\mu/2) \cos (\mu/2) \cos \alpha) \\
 c_{33} &= 1 - 2 \sin^2 (\mu/2) \sin^2 \gamma.
 \end{aligned}$$

If the following substitutions are made,

$$\begin{aligned}
 \zeta &= \cos \alpha \sin (\mu/2) \\
 \eta &= \cos \beta \sin (\mu/2) \\
 \delta &= \cos \gamma \sin (\mu/2) \\
 \chi &= \cos (\mu/2),
 \end{aligned}$$

the matrix of (III-46) becomes

$$\begin{aligned}
 c_{11} &= \zeta^2 - \eta^2 - \delta^2 + \chi^2 \\
 c_{12} &= 2(\zeta \eta + \delta \chi) \\
 c_{13} &= 2(\zeta \delta + \eta \chi)
 \end{aligned}$$

$$\begin{aligned}
c_{21} &= 2(\xi\eta - \delta\chi) \\
c_{22} &= \xi^2 + \eta^2 - \delta^2 + \chi^2 \\
c_{23} &= 2(\eta\delta + \xi\chi) \\
c_{31} &= 2(\xi\delta + \eta\chi) \\
c_{32} &= 2(\eta\delta + \xi\chi) \\
c_{33} &= -\xi^2 - \eta^2 + \delta^2 + \chi^2.
\end{aligned}
\tag{III-47}$$

These four parameters are called the Euler parameter. It may be seen that they obey the relationship

$$\xi^2 + \eta^2 + \delta^2 + \chi^2 = 1. \tag{III-48}$$

Cayley-Klein Parameters

In the Cayley-Klein development of the four-parameter system, it is found that a 2×2 complex matrix may be used to represent a real rotation, rather than a 3×3 real matrix. Consider such a matrix (H),

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}. \tag{III-49}$$

The requirement placed on this matrix is that it be unitary, which is to say that the product of (H) and its conjugate transpose must yield the unit matrix. In addition, it is required that the determinant of the matrix (H) have the value +1. The unitary condition allows ± 1

for the determinant, so this is an additional requirement. The unitary condition may be written as

$$\begin{bmatrix} h_{11}^* & h_{21}^* \\ h_{12}^* & h_{22}^* \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (\text{III-50})$$

Expanding and equating components gives

$$\begin{aligned} h_{11}^* h_{11} + h_{21}^* h_{21} &= 1, \\ h_{11}^* h_{12} + h_{21}^* h_{22} &= 0, \\ h_{12}^* h_{11} + h_{22}^* h_{21} &= 0, \\ h_{12}^* h_{12} + h_{22}^* h_{22} &= 1. \end{aligned} \quad (\text{III-51})$$

The second and third equations are the same, being merely complex conjugates of each other. The first and fourth equations have no imaginary component, whereas the second (or third) has both real and imaginary parts. Therefore, the three independent equations contain four conditions. These, together with the determinant requirement that $h_{11}h_{22} - h_{21}h_{12} = +1$, make it possible to determine certain relationships among the four quantities h_{mn} . It may be shown that $h_{22} = h_{11}^*$, and that $h_{21} = -h_{12}^*$, and thus the matrix may be written as

$$H = \begin{bmatrix} h_{11} & h_{12} \\ -h_{12}^* & h_{11}^* \end{bmatrix}. \quad (\text{III-52})$$

The quantities h_{11} , h_{12} , h_{22} are usually referred to as the Cayley-Klein parameters. It will be noted that they are complex numbers.

While it is convenient to use them as such in analytical operations (and this is the purpose for which Klein developed them), a physical computer must treat complex numbers in terms of their real and imaginary parts. Therefore, it is convenient to introduce four other quantities defined as follows:

$$\begin{aligned} h_{11} &= e_1 + ie_2 \\ h_{12} &= e_3 + ie_4 \end{aligned} \quad (\text{III-53})$$

where the e 's are all real numbers, and i is the square root of -1 .

Using these definitions, the matrix (H) may be written as

$$H = \begin{bmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{bmatrix}. \quad (\text{III-54})$$

Now consider another complex matrix (P) , which has the form

$$P = \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix} \quad (\text{III-55})$$

where x , y , and z are real numbers. It will be noted that the matrix (P) is equal to its own conjugate transpose, therefore, it is Hermitian. Now consider a transformation of (P) of the form

$$P' = (H)(P)(H)^+ \quad (\text{III-56})$$

where $(H)^+$ designates the conjugate transpose of (H) . Since (H) is unitary, $(H)^+ = (H)^I$, so (III-56) is

$$P' = (H)(P)(H)^I \quad (\text{III-57})$$

This is a similarity transformation. It can be shown that the Hermitian property and the trace are both invariant under a similarity transformation. Therefore, the transformed matrix $(P)'$ must have the form

$$P' = \begin{bmatrix} z' & x - iy' \\ x' + iy' & -z' \end{bmatrix}. \quad (\text{III-58})$$

The fact that the determinant of (P) must equal the determinant of $(P)'$ gives

$$\bar{x}^2 + y^2 + z^2 = \bar{x}'^2 + y'^2 + z'^2. \quad (\text{III-59})$$

If x , y , and z are viewed as components of a vector, then (III-59) is the requirement that the length of the vector remain unchanged.

(III-57) may be written

$$\begin{bmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{bmatrix} = \begin{bmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{bmatrix} \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix} \\ \begin{bmatrix} e_1 - ie_2 & -e_3 - ie_4 \\ e_3 - ie_4 & e_1 + ie_2 \end{bmatrix} \quad (\text{III-60})$$

If the operations of (III-60) are carried out, it is found that

$$\begin{aligned} x' &= (e_1^2 - e_2^2 - e_3^2 + e_4^2)x - 2(e_1e_2 + e_3e_4)y + 2(e_2e_4 - e_1e_3)z, \\ y' &= 2(e_3e_4 - e_1e_2)x + (e_1^2 - e_2^2 + e_3^2 - e_4^2)y + 2(e_2e_3 + e_1e_4)z, \\ z' &= 2(e_1e_3 + e_2e_4)x + 2(e_2e_3 - e_1e_4)y + (e_1^2 + e_2^2 - e_3^2 - e_4^2)z. \end{aligned} \quad (\text{III-61})$$

These equations represent a linear transformation between the components of x , y , and z , and the components of x' , y' , and z' .

The matrix for this transformation is

$$C = \begin{bmatrix} (e_1^2 - e_2^2 - e_3^2 + e_4^2) & 2(e_1e_2 + e_3e_4) & 2(e_2e_4 - e_1e_3) \\ 2(e_3e_4 - e_1e_2) & (e_1^2 - e_2^2 + e_3^2 - e_4^2) & 2(e_2e_3 + e_1e_4) \\ 2(e_1e_3 + e_2e_4) & 2(e_2e_3 - e_1e_4) & (e_1^2 + e_2^2 - e_3^2 - e_4^2) \end{bmatrix} \cdot \quad (III-62)$$

It may be shown directly that this matrix satisfies the orthogonality conditions, but it is also proven from (III-59). (III-60) shows that the nine direction cosines may be expressed in terms of the four e 's. If (III-53) is substituted into (III-51) it is found that

$$e_1^2 + e_2^2 + e_3^2 + e_4^2 = 1, \quad (III-63)$$

and therefore, only three of the e 's are independent. The identity of these four quantities with the Euler parameters is obvious.

Comparison of (III-40) and (III-45) gives

$$e_1 = \chi, \quad e_2 = \delta, \quad e_3 = \eta, \quad e_4 = \zeta. \quad (III-64)$$

It is also possible to view this process as two successive rotations in terms of the e 's themselves. Consider one rotation defined by e_1 , e_2 , e_3 , and e_4 . After this, another rotation is performed which is described by e_1' , e_2' , e_3' , and e_4' . There is

some set of e's called e_1'' , e_2'' , e_3'' , and e_4'' which describes the final orientation after the two rotations. This combined set may be found by multiplying the (H) matrices of the two rotations in the correct sequence. The equation is

$$H'' = \begin{bmatrix} e_1'' + ie_2'' & e_3'' + ie_4'' \\ -e_3'' + ie_4'' & e_1'' - ie_2'' \end{bmatrix} = \begin{bmatrix} e_1' + ie_2' & e_3' + ie_4' \\ -e_3' + ie_4' & e_1' - ie_2' \end{bmatrix} \begin{bmatrix} e_1' + ie_2' & e_3' + ie_4' \\ -e_3' - ie_4' & e_1' - ie_2' \end{bmatrix} \quad (\text{III-65})$$

Expanding this equation and equating components gives

$$\begin{aligned} e_1'' &= e_1'e_1 - e_2'e_2 - e_3'e_3 - e_4'e_4, \\ e_2'' &= e_2'e_1 + e_2'e_1 + e_3'e_4 - e_4'e_3, \\ e_3'' &= e_1'e_3 - e_2'e_4 + e_3'e_1 + e_4'e_2, \\ e_4'' &= e_2'e_3 + e_1'e_4 + e_4'e_1 - e_3'e_2, \end{aligned} \quad (\text{III-66})$$

By use of these equations, successive transformations may be handled in terms of the e's directly.

Quaternions

The most brilliant formulation of the four parameters was made by Hamilton in 1843. He developed a new type of entity called a "quaternion". It is composed of four parts,

$$q = S + ia + jb + kc, \quad (\text{III-67})$$

where S, a, b, and c are real numbers, and the indices i, j, and k are

defined by the following rules;

$$\begin{aligned} i^2 &= -1, & ij &= -ji = k, \\ j^2 &= -1, & jk &= -kj = i, \\ k^2 &= -1, & ki &= -ik = j. \end{aligned}$$

The conjugate of the quaternion q is

$$q^* = S - ia - jb - kc. \quad (\text{III-68})$$

Using the laws for the indices quoted above, it may be easily shown that

$$qq^* = q^*q = S^2 + a^2 + b^2 + c^2, \quad (\text{III-69})$$

which is called the length or norm of the quaternion. If this norm is unity, then a special form of quaternion results, a versor. It is possible to make use of versors to describe a coordinate transformation. The quantity S is called the real or scalar part of the quaternion. Consider V a vector of components X , Y , and Z :

$$V = iX + jY + kZ. \quad (\text{III-70})$$

Examine the operation

$$V' = q^*Vq \quad (\text{III-71})$$

where q is a versor. So far there is no particular reason to expect that V' will be a vector, but this turns out to be the case. (III-71) may be written as:

$$V' = (S - ia - jb - kc)(iX + jY + kZ)(S + ia + jb + kc) \quad (\text{III-72})$$

When this equation is expanded, making use of the rules for indices, the result is

$$\begin{aligned}
 V' = & i \left[X(S^2 + a^2 - b^2 - c^2) + Y(2Sc + 2ab) + Z(2ac - 2Sb) \right] + \\
 & j \left[X(2ab - 2Sc) + Y(S^2 - a^2 + b^2 - c^2) + Z(2Sa + 2cb) \right] + \\
 & k \left[X(2Sb + 2ac) + Y(2bc - 2Sa) + Z(S^2 - a^2 - b^2 + c^2) \right]
 \end{aligned}
 \tag{III-73}$$

This is simply a coordinate transformation whose transformation matrix is

$$\begin{bmatrix}
 S^2 + a^2 - b^2 - c^2 & 2(Sc + ab) & 2(ac - Sb) \\
 2(ab - Sc) & S^2 - a^2 + b^2 - c^2 & 2(Sa + cb) \\
 2(Sb + ac) & 2(bc - Sa) & S^2 - a^2 - b^2 + c^2
 \end{bmatrix}
 \tag{III-74}$$

The correlations with matrices derived in the preceding sections are

$$S = e_1 = \chi, \quad c = e_2 = \delta, \quad b = e_3 = \eta, \quad a = e_4 = \zeta.
 \tag{III-75}$$

The matter of two successive rotations may be handled quite easily.

Assume a transformed vector with the versor q_1 ,

$$V' = q_1^* V q_1.
 \tag{III-76}$$

Apply the versor q_2 ,

$$V'' = q_2^* V' q_2 = q_2^* q_1^* V q_1 q_2.
 \tag{III-77}$$

New vectors are defined as $q_4 = q_2^* q_1^*$ and $q_3 = q_1 q_2$. It may be seen

that the relationship between q_3 and q_4 is given as

$$q_2^* q_1^* = q_4 \quad (\text{III-78})$$

$$\text{and } q_2 q_2^* q_1^* = q_2 q_4$$

and since q_2 is a versor, $q_2 q_2^* = 1$. Therefore, (III-78) reduces to

$$q_1^* = q_2 q_4. \quad (\text{III-79})$$

Now q_1 is applied on the left,

$$q_1 q_1^* = q_1 q_2 q_4 = 1 = q_3 q_4, \quad (\text{III-80})$$

so that q_4 must equal the conjugate of q_3 . This means that

$$V'' = q_3^* V q_3. \quad (\text{III-81})$$

The equation $q_3 = q_1 q_2$ may now be written as

$$S_3 + ia_3 + jb_3 + kc_3 = (S_1 + ia_1 + jb_1 + kc_1)(S_2 + ia_2 + jb_2 + kc_2).$$

Expanding this equation and equating like components gives:

$$\begin{aligned} S_3 &= S_1 S_2 - a_1 a_2 - b_1 b_2 - c_1 c_2, \\ a_3 &= S_1 a_2 + S_2 a_1 + b_1 c_2 - c_1 b_2, \\ b_3 &= S_1 b_2 - a_1 c_2 + b_1 S_2 + c_1 a_2, \\ c_3 &= S_1 c_2 + a_1 b_2 - b_1 a_2 + c_1 S_2. \end{aligned} \quad (\text{III-82})$$

These equations are identical with (III-66) which was developed in the same connection by use of the Cayley-Klein parameters. Thus, the quaternion method leads to the same results as did the preceding developments.

Infinitesimal Rotations

The primary interest is in determining the orientation from the rate of rotation through a process of integration. Accordingly, it is necessary to relate the rates of change of the four parameters to the rates of rotation of the axis system.

It has been shown that an orthogonal transformation may be represented by a complex matrix having certain properties. It is now of interest to investigate this matrix when an infinitesimal rotation is performed. First, assume that this infinitesimal rotation consists of a rotation through the angle $\Delta \mu$ about a line which makes angles of α , β , and γ with the X, Y, and Z axes respectively. Recall that the matrix (H) may be expressed

$$H = \begin{bmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{bmatrix} \quad (III-83)$$

Applying the geometrical interpretation of the e's from (III-64) gives

$$H = \begin{bmatrix} \cos(\Delta\mu/2) + i \cos\gamma \sin(\Delta\mu/2) & \cos\beta \sin(\Delta\mu/2) + i \cos\alpha \sin(\mu/2) \\ -\cos\beta \sin(\Delta\mu/2) + i \cos\alpha \sin(\Delta\mu/2) & \cos(\Delta\mu/2) - i \cos\gamma \sin(\Delta\mu/2) \end{bmatrix} \quad (III-84)$$

From this, it can be seen that the infinitesimal rotation may be represented by

$$H_{\epsilon} = \begin{bmatrix} 1 + i (\Delta\mu/2) \cos\gamma & (\Delta\mu/2) \cos\beta + i (\Delta\mu/2) \cos\alpha \\ -(\Delta\mu/2) \cos\beta + i (\Delta\mu/2) \cos\alpha & 1 - i (\Delta\mu/2) \cos\gamma \end{bmatrix} \quad (III-85)$$

since $\cos(\Delta\mu/2) \approx 1$, and since $\sin(\Delta\mu/2) \approx \Delta\mu/2$.

It is expected that any matrix representing an infinitesimal rotation will differ only slightly from the identity matrix. This is true of the above matrix, and it may be shown more clearly by writing it as follows:

$$H_{\epsilon} = I + (\epsilon)$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$(\epsilon) = \frac{\Delta u}{2} \begin{bmatrix} i \cos \gamma & \cos \beta + i \cos \alpha \\ -\cos \beta + i \cos \alpha & -i \cos \gamma \end{bmatrix}. \quad (\text{III-86})$$

Now assume that this infinitesimal rotation takes place during a small time interval, and that $(H)'$ is the matrix at the end of the interval. Then the time derivative of (H) may be written as

$$\frac{d(H)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{(H)' - (H)}{\Delta t}. \quad (\text{III-87})$$

The final matrix $(H)'$ may also be viewed as the result of two rotations, first (H) and then $(H)_{\epsilon}$. In other words, $(H)' = (H)_{\epsilon} (H)$. The insertion of this value into the above equation gives

$$\frac{d(H)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{(\epsilon)}{\Delta t} (H). \quad (\text{III-88})$$

Since (H) is not affected by the time increment, the limiting process refers only to the quantity $\frac{(\epsilon)}{\Delta t}$.

$$\frac{(\epsilon)}{\Delta t} = \frac{1}{2} \frac{\Delta \mu}{\Delta t} \begin{bmatrix} i \cos \gamma & \cos \beta + i \cos \alpha \\ -\cos \beta + i \cos \alpha & -i \cos \gamma \end{bmatrix} \quad (\text{III-89})$$

In the limit, the quantity $\frac{\Delta \mu}{\Delta t}$ is simply the scalar magnitude of the angular velocity vector. If $\frac{d\phi_x}{dt}$, $\frac{d\phi_y}{dt}$, and $\frac{d\phi_z}{dt}$ are the components of this velocity vector along the X, Y, and Z axes, then evidently $\frac{d\mu}{dt} \cos \alpha = \frac{d\phi_x}{dt}$, $\frac{d\mu}{dt} \cos \beta = \frac{d\phi_y}{dt}$, and $\frac{d\mu}{dt} \cos \gamma = \frac{d\phi_z}{dt}$ so that

$$\lim_{\Delta t \rightarrow 0} \frac{(\epsilon)}{\Delta t} = \frac{1}{2} \begin{bmatrix} i \frac{d\phi_z}{dt} & \frac{d\phi_y}{dt} + i \frac{d\phi_x}{dt} \\ -\frac{d\phi_y}{dt} + i \frac{d\phi_x}{dt} & -i \frac{d\phi_z}{dt} \end{bmatrix} \quad (\text{III-90})$$

Therefore,

$$\frac{d(H)}{dt} = \frac{1}{2} \begin{bmatrix} i \frac{d\phi_z}{dt} & \frac{d\phi_y}{dt} + i \frac{d\phi_x}{dt} \\ -\frac{d\phi_y}{dt} + i \frac{d\phi_x}{dt} & -i \frac{d\phi_z}{dt} \end{bmatrix} (H) \quad (\text{III-91})$$

It is also possible to show, by a straight forward limiting process, that the time derivative of a matrix is also a matrix whose elements are the time derivatives of the elements of the original matrix.

Therefore,

$$\begin{bmatrix} \dot{e}_1 + i\dot{e}_2 & \dot{e}_3 + i\dot{e}_4 \\ -\dot{e}_3 + i\dot{e}_4 & \dot{e}_1 - i\dot{e}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} i \frac{d\phi_z}{dt} & \frac{d\phi_y}{dt} + i \frac{d\phi_x}{dt} \\ -\frac{d\phi_y}{dt} + i \frac{d\phi_x}{dt} & -i \frac{d\phi_z}{dt} \end{bmatrix} \begin{bmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{bmatrix} \quad (\text{III-92})$$

Expanding and equating like terms yields

$$2\dot{e}_1 = -e_4 \frac{d\phi_x}{dt} - e_3 \frac{d\phi_y}{dt} - e_2 \frac{d\phi_z}{dt} ,$$

$$2\dot{e}_2 = -e_3 \frac{d\phi_x}{dt} + e_4 \frac{d\phi_y}{dt} + e_1 \frac{d\phi_z}{dt} ,$$

$$2\dot{e}_3 = +e_2 \frac{d\phi_x}{dt} + e_1 \frac{d\phi_y}{dt} - e_4 \frac{d\phi_z}{dt} ,$$

$$2\dot{e}_4 = +e_1 \frac{d\phi_x}{dt} - e_2 \frac{d\phi_y}{dt} + e_3 \frac{d\phi_z}{dt} .$$

(III-93)

These are the equations which are used to compute the four parameters in an actual simulation.

APPENDIX A

MATRIX OPERATIONS AND DEFINITIONS^{19,20,21,22}

A system of m nonhomogeneous linear equations in the n unknowns x_1, x_2, \dots, x_n is

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = y_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = y_2$$

(A-1)

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = y_m$$

This system of equations may be represented by the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad (A-2)$$

An abbreviated symbolism for (A-2) is

$$AX = Y ; A = (a_{ij})_{m,n} , \quad (A-3)$$

which may also be written as

$$y_i = \sum_{j=1}^n a_{ij} x_j \quad i = 1, 2, \dots, m.$$

Definition 1.1

A matrix is an ordered array of $m \times n$ scalars a_{ij} arranged in m rows and n columns. The scalar a_{ij} in row i column j is called the ij entry of A . If $i = j$, then a_{ij} lies on the diagonal; if $i \neq j$, then a_{ij} is either above ($i < j$, superdiagonal) or below ($i > j$, subdiagonal) the diagonal.

Definition 1.2

The trace or spur of a matrix is the sum of the diagonal entries.

$$\text{trace}(A) = \text{spur}(A) = \sum_{j=1}^n a_{jj}.$$

Definition 1.3

A square matrix is a matrix with $m = n$.

A row vector is a matrix with $m = 1$.

A column vector is a matrix with $n = 1$.

Examples:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

SQUARE MATRIX

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$$

ROW VECTOR

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$$

COLUMN VECTOR

Definition 1.4

A null matrix is one in which all entries are zero.

Definition 1.5

A matrix is called

- | | | | |
|------------------------------|-----------------|-----|-----------------|
| 1. <u>Diagonal</u> , | if $a_{ij} = 0$ | for | $i \neq j$ |
| 2. <u>Superdiagonal</u> , | if $a_{ij} = 0$ | for | $i \geq j$ |
| 3. <u>Subdiagonal</u> , | if $a_{ij} = 0$ | for | $i \leq j$ |
| 4. <u>Upper-triangular</u> , | if $a_{ij} = 0$ | for | $i > j$ |
| 5. <u>Lower-triangular</u> , | if $a_{ij} = 0$ | for | $i < j$ |
| 6. <u>Tridiagonal</u> , | if $a_{ij} = 0$ | for | $ i - j > 1$. |

Definition 1.6

The identity matrix or unit matrix (I) is a diagonal matrix with its diagonal entries equal to one.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (\delta_{ij})$$

where δ_{ij} is the Kronecker delta symbol and

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Definition 1.7

The transpose of matrix A, (A^T) , is the matrix obtained from A by interchanging rows and columns. (The first row becomes the first column, the first column becomes the first row, etc.)

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Definition 1.8

If the entries in matrix A are complex, then the conjugate transpose of A , (A^{CT}) , is the transpose of A with the sign of the imaginary part of all entries changed.

Rule 1.1 Matrix Addition

Matrix A is added to matrix B to give the sum, matrix C , by adding the a_{ij} entry to the b_{ij} entry to give the c_{ij} entry; that is, $c_{ij} = a_{ij} + b_{ij}$. It is obvious that matrix addition is defined only for matrices of the same dimensions.

Rule 1.2 Matrix Addition

The distributive laws hold for matrix addition. If d is a scalar then

$$d(A + B) = dA + dB$$

and

$$(d_1 + d_2)A = d_1A + d_2A.$$

Rule 1.3 Matrix Multiplication

To multiply two matrices together to obtain the third; multiply the elements of row i of the left factor by corresponding elements of column k of the right factor and add to obtain the ij entry of the product, i.e., if $AB = C$, then

$$c_{ij} = \sum_{n=1}^m a_{in} b_{nj}.$$

The product AB is defined only when A has as many columns as B has rows. Then AB has as many rows as A and as many columns as B .

Rule 1.4 Matrix Multiplication

The distributive law holds for matrix multiplication. Therefore,

$$A(B + C) = AB + AC$$

and

$$(A + B)D = AD + BD$$

whenever the products are defined.

Rule 1.5 Matrix Multiplication

The commutative law $AB = BA$ holds only in special cases, but does not hold in general.

Definition 1.9

If X and Y are $n \times 1$ column vectors, and X^T and Y^T the corresponding $1 \times n$ row vector, the product

$$X^T Y = Y^T X = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

is a scalar called the scalar product of X and Y . The related quantity

$$X^{CT} Y = \overline{(Y^{CT} X)} = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \dots + \bar{x}_n y_n$$

is called the complex scalar product of X and Y .

Definition 1.10

The length, or norm $|X|$, of a vector $X \neq 0$ is the positive quantity $|X| = X^{CT}X > 0$. If $X = 0$, $|0| = 0$ and X is a null vector.

A vector of unit length is called a unit vector.

Definition 1.11

The matrix A is called idempotent if $A^2 = A$.

Definition 1.12

A matrix A is said to have an inverse, (A^I or A^{-1}), if $AA^I = I$.

Definition 1.13

A matrix A is said to be invertible or nonsingular if it has an inverse, singular if it does not.

Definition 1.14

Two column vectors X and Y are called orthogonal if $X^TY = 0$; complex orthogonal if $X^{CT}Y = 0$.

Definition 1.15

If a matrix X is such that $X^T = X^I$, then X is orthogonal.

Definition 1.16

The matrix A is called normal if $AA^{CT} = A^{CT}A$.

Definition 1.17

The matrix A is called hermitian if $A^{CT} = A$.

Definition 1.18

The matrix A is called unitary if $AA^{CT} = I$, that is if $A^{CT} = A^I$.

Definition 1.19

Any transformation of a matrix having the form

$$A' = BAB^I$$

is known as a similarity transformation and the
 $\text{trace}(A') = \text{trace}(A)$.

Definition 1.20

A square matrix A has associated with it a scalar quantity called the determinant of the matrix, and is denoted by $\det(A)$ or $|A|$.

$$\det(A) = \sum_{i=1}^n a_{ij} \nabla_{ij} = \sum_{j=1}^n a_{ij} \nabla_{ij}$$

where ∇_{ij} is the cofactor of the ij^{th} element of A as defined in

Definition 1.22.

Definition 1.21

The determinant of the $(n-1) \times (n-1)$ matrix obtained from a $n \times n$ matrix by crossing out the i^{th} row and j^{th} column is called the minor of the ij^{th} element and is indicated by M_{ij} .

Definition 1.22

The cofactor of the ij^{th} element is the minor of the ij^{th} element multiplied by $(-1)^{i+j}$; i.e., $\nabla_{ij} = (-1)^{i+j} M_{ij}$. The cofactor is a signed minor.

Definition 1.23

The adjoint of matrix A , $\text{adj}(A)$, is the transpose of the matrix obtained by replacing each term of A by its cofactor.

Definition 1.24

The inverse of A is obtained by dividing each term in the adjoint of A by the determinant of A .

$$A^{-1} = A^{-1} = \frac{\text{adj}(A)}{|A|}.$$

APPENDIX B
NUMERICAL INTEGRATION ^{23,24}

"Numerical integration is a process of computing the value of a definite integral from a set of numerical values of the integrand." ²³

A list of the most commonly used integrating techniques along with a short explanation of each is presented in this section. Proofs and derivations of the methods given can be found in the references listed at the end of this report in the Selected Bibliography.

Rectangular Rule

When the rectangular rule is used for numerical integration, the function is approximated by a staircase curve made up of a set of constant step functions. In graphical terms the area under the curve over the interval, Δt , is assumed to be equal to the area of a rectangle. If the interval is very small, this numerical integration scheme is quite good.

To apply the rectangular rule the derivative of the function, \dot{a}_n , must be known where a_n is the value of the integral at time $n\Delta t$, and

$$a_n = \int_{(n-1)\Delta t}^{n\Delta t} \dot{a}_n dt + a_{n-1} = \Delta t(\dot{a}_n) + a_{n-1}. \quad (B-1)$$

Newton's Formula

When Newton's formula is used for numerical integration, the function is approximated over each interval by a suitable polynomial. Newton formulated the following general formula for obtaining this polynomial:

$$\begin{aligned}
 a_n - a_0 &= \int_{t_0}^{t_0 + n\Delta t} a \, dt = \Delta t \, na_0 + \frac{n^2}{2} \Delta a_0 \\
 &+ \left(\frac{n^3}{3} - \frac{n^2}{2} \right) \frac{\Delta^2 a_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2 \right) \frac{\Delta^3 a_0}{3!} \\
 &+ \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2 \right) \frac{\Delta^4 a_0}{4!} \\
 &+ \left(\frac{n^6}{6} - 2n^5 + \frac{35n^4}{4} - \frac{50n^3}{3} + 12n^2 \right) \frac{\Delta^5 a_0}{5!} \\
 &+ \left(\frac{n^7}{7} - \frac{15n^6}{6} + 17n^5 - \frac{225}{4}n^4 + \frac{274}{3}n^3 - 60n^2 \right) \frac{\Delta^6 a_0}{6!}, \quad (B-2)
 \end{aligned}$$

where

$$\Delta a_0 = a_1 - a_0$$

$$\Delta^2 a_0 = a_2 - 2a_1 + a_0$$

$$\Delta^3 a_0 = a_3 - 3a_2 + 3a_1 - a_0$$

$$\Delta^4 a_0 = a_4 - 4a_3 + 6a_2 - 4a_1 + a_0$$

and so forth.

By letting $n = 1, 2, \dots$ in (B-2), and by neglecting the terms higher than n , various quadrature formulas can be obtained.

Letting $n = 1$ gives the trapezoidal rule:

$$a_n = a_{n-1} + \frac{\Delta t}{2} (a_n + a_{n-1}) . \quad (\text{B-3})$$

Letting $n = 2$ gives Simpson's rule:

$$a_n = a_{n-2} + \frac{\Delta t}{3} (a_n + 4a_{n-1} + a_{n-2}) . \quad (\text{B-4})$$

Letting $n = 6$ gives Weddle's rule:

$$a_n = a_{n-6} + \frac{3\Delta t}{10} (a_n + 5a_{n-1} + a_{n-2} + 6a_{n-3} + a_{n-4} + 5a_{n-5} + a_{n-6}) . \quad (\text{B-5})$$

Method of Weighted Averages

A good simple method of numerical integration is the weighted average method. This method is based upon the assumption that a function changes the same amount in the following interval as it does in the previous interval. Therefore,

$$a_{n+1} - a_n = a_n - a_{n-1}$$

or

$$a_{n+1} = 2a_n - a_{n-1} \quad (\text{B-6})$$

and the average

$$a = \frac{a_{n+1} + a_n}{2} = \frac{3a_n - a_{n-1}}{2} \quad (\text{B-7})$$

so that

$$a_n = a_{n-1} + \frac{\Delta t}{2} (3a_n - a_{n-1}) . \quad (\text{B-8})$$

Runge-Kutta Method

The Runge-Kutta method is essentially a refinement of the averaging techniques. Nearly every text on numerical analysis has a section on Runge-Kutta methods, and a discussion of how they are derived for different order approximations. The most common one is the fourth-order approximation. Consider the differential equation:

$$\frac{dy}{dt} = \dot{y} = f(t, y) \quad (\text{B-9})$$

with $y = y_0$ at $t = t_0$. The increment for advancing y is given by

$$\Delta y = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \quad (\text{B-10})$$

where

$$\begin{aligned} k_1 &= \Delta t f(t_0, y_0), \\ k_2 &= \Delta t f\left(t_0 + \frac{\Delta t}{2}, y_0 + \frac{k_1}{2}\right), \\ k_3 &= \Delta t f\left(t_0 + \frac{\Delta t}{2}, y_0 + \frac{k_2}{2}\right), \\ k_4 &= \Delta t f(t_0 + \Delta t, y_0 + k_3). \end{aligned}$$

The new values of t and y at the end of the interval Δt are

$$t_1 = t_0 + \Delta t \quad (\text{B-11})$$

$$y_1 = y_0 + \Delta y. \quad (\text{B-12})$$

The Runge-Kutta fourth order approximation applied to simultaneous equations is as follows.

If

$$\frac{dx}{dt} = \dot{x} = f_1(t, x, y) \quad (\text{B-13})$$

and

$$\frac{dy}{dt} = \dot{y} = f_2(t, x, y) \quad (\text{B-14})$$

then

$$\begin{aligned} k_1 &= \Delta t f_1(t_o, x_o, y_o), \\ k_2 &= \Delta t f_1(t_o + \frac{\Delta t}{2}, x_o + \frac{k_1}{2}, y_o + \frac{l_1}{2}), \\ k_3 &= \Delta t f_1(t_o + \frac{\Delta t}{2}, x_o + \frac{k_2}{2}, y_o + \frac{l_2}{2}), \\ k_4 &= \Delta t f_1(t_o + \Delta t, x_o + k_3, y_o + l_3), \\ l_1 &= \Delta t f_2(t_o, x_o, y_o), \\ l_2 &= \Delta t f_2(t_o + \frac{\Delta t}{2}, x_o + \frac{k_1}{2}, y_o + \frac{l_1}{2}), \\ l_3 &= \Delta t f_2(t_o + \frac{\Delta t}{2}, x_o + \frac{k_2}{2}, y_o + \frac{l_2}{2}), \\ l_4 &= \Delta t f_2(t_o + \Delta t, x_o + k_3, y_o + l_3). \end{aligned} \quad (\text{B-15})$$

The new values of x and y at the end of the interval Δt are

$$x_1 = x_o + \Delta x \quad (\text{B-16})$$

$$y_1 = y_o + \Delta y \quad (\text{B-17})$$

where

$$\begin{aligned} \Delta x &= \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\ \Delta y &= \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4). \end{aligned}$$

Milne's Method

Milne's method is derived from Newton's forward interpolation formula. Two formulas are used; one for integrating ahead by **extra-**polation, and the other for checking the extrapolated value. The two

formulas are given below.

$$a_{n+1}^{(1)} = a_{n-3} + \frac{4\Delta t}{3}(2a_{n-2} - a_{n-1} + 2a_n). \quad (\text{B-18})$$

$$a_{n+1}^{(2)} = a_{n-1} + \frac{\Delta t}{3}(a_{n-1} + 4a_n + a_{n+1}). \quad (\text{B-19})$$

The first equation is used to integrate ahead by extrapolation and the other is used to check the extrapolated value. This simple formula provides a check on the accuracy of each computation. If the value of

$$E = a_{n+1}^{(1)} - a_{n+1}^{(2)} \quad (\text{B-20})$$

becomes erratic, and no mistake has been made, then a smaller Δt is in order.

APPENDIX C

ORTHOGONAL MATRIX

The rotational matrix that transforms the measured velocities in the vehicle coordinate system to the inertial coordinate system is a proper rotational matrix, and, therefore, is orthogonal. If the rotational matrix

$$(C) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \quad (C-1)$$

is orthogonal, then

$$c_{11}^2 + c_{21}^2 + c_{31}^2 = 1$$

$$c_{12}^2 + c_{22}^2 + c_{32}^2 = 1$$

$$c_{13}^2 + c_{23}^2 + c_{33}^2 = 1$$

(C-2)

$$c_{11}^2 + c_{12}^2 + c_{13}^2 = 1$$

$$c_{21}^2 + c_{22}^2 + c_{23}^2 = 1$$

$$c_{31}^2 + c_{32}^2 + c_{33}^2 = 1$$

are the six normality conditions and

$$c_{11}c_{21} + c_{12}c_{22} + c_{12}c_{23} = 0$$

$$c_{11}c_{31} + c_{12}c_{32} + c_{13}c_{33} = 0$$

$$c_{21}c_{31} + c_{22}c_{32} + c_{23}c_{33} = 0$$

$$c_{11}c_{12} + c_{21}c_{22} + c_{31}c_{32} = 0$$

(C-3)

$$c_{11}c_{13} + c_{21}c_{23} + c_{31}c_{33} = 0$$

$$c_{12}c_{13} + c_{22}c_{23} + c_{32}c_{33} = 0$$

are the six orthogonality conditions which must be satisfied. Nine other relationships that must be satisfied may be obtained and are

$$c_{11} = c_{22}c_{33} - c_{23}c_{32}$$

$$c_{12} = c_{23}c_{31} - c_{21}c_{33}$$

$$c_{13} = c_{21}c_{32} - c_{22}c_{31}$$

$$c_{21} = c_{13}c_{32} - c_{12}c_{33}$$

$$c_{22} = c_{11}c_{33} - c_{13}c_{31}$$

$$c_{23} = c_{12}c_{31} - c_{11}c_{32}$$

$$c_{31} = c_{12}c_{23} - c_{13}c_{22}$$

$$c_{32} = c_{13}c_{21} - c_{11}c_{23}$$

(C-4)

$$c_{33} = c_{11}c_{22} - c_{12}c_{21}.$$

APPENDIX D
COORDINATE SYSTEMS²⁵

The space-fixed coordinate system is shown in Figure D-1. The X_s axis lies in the plane defined by the polar axis of the earth and the launching site. X_s axis is parallel to the local gravity vector at the launch site, and is directed from the center of the earth towards the surface near the launch site. The Z_s axis is perpendicular to the X_s axis, is parallel to the aiming azimuth, and is positive down range. The Y_s axis completes a standard right-handed coordinate system.

The vehicle-fixed coordinate system is shown in Figure D-2. The origin of this system is located at the center of mass of the vehicle. The X_v axis is directed along the longitudinal axis of the vehicle and is positive in the nominal direction of positive thrust acceleration. The Z_v axis is perpendicular to X_v and is defined by "Position 1", a predesignated position on the vehicle. The Y_v axis completes a standard right-handed coordinate system.

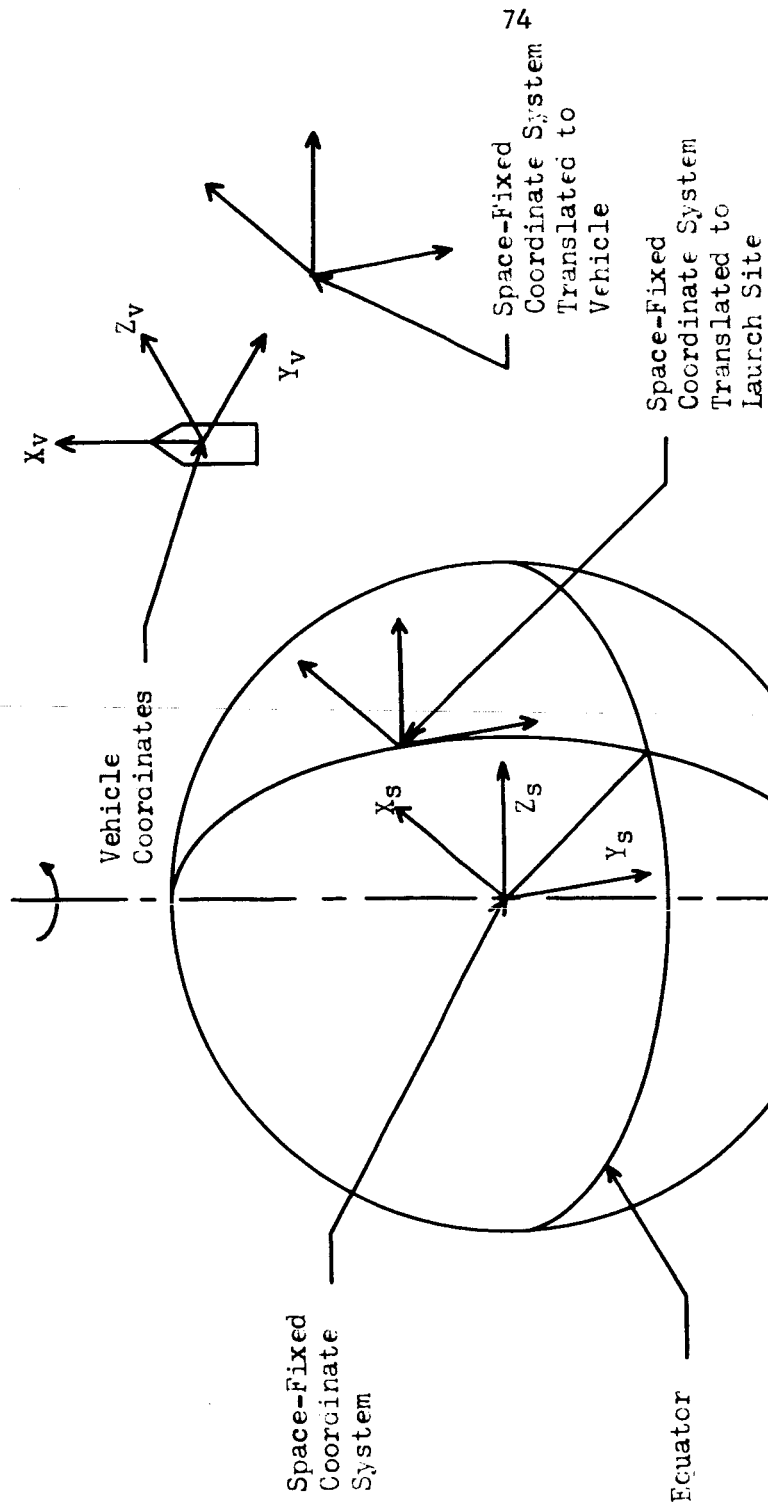


Fig. D-1-1--Space-Fixed Coordinate System.

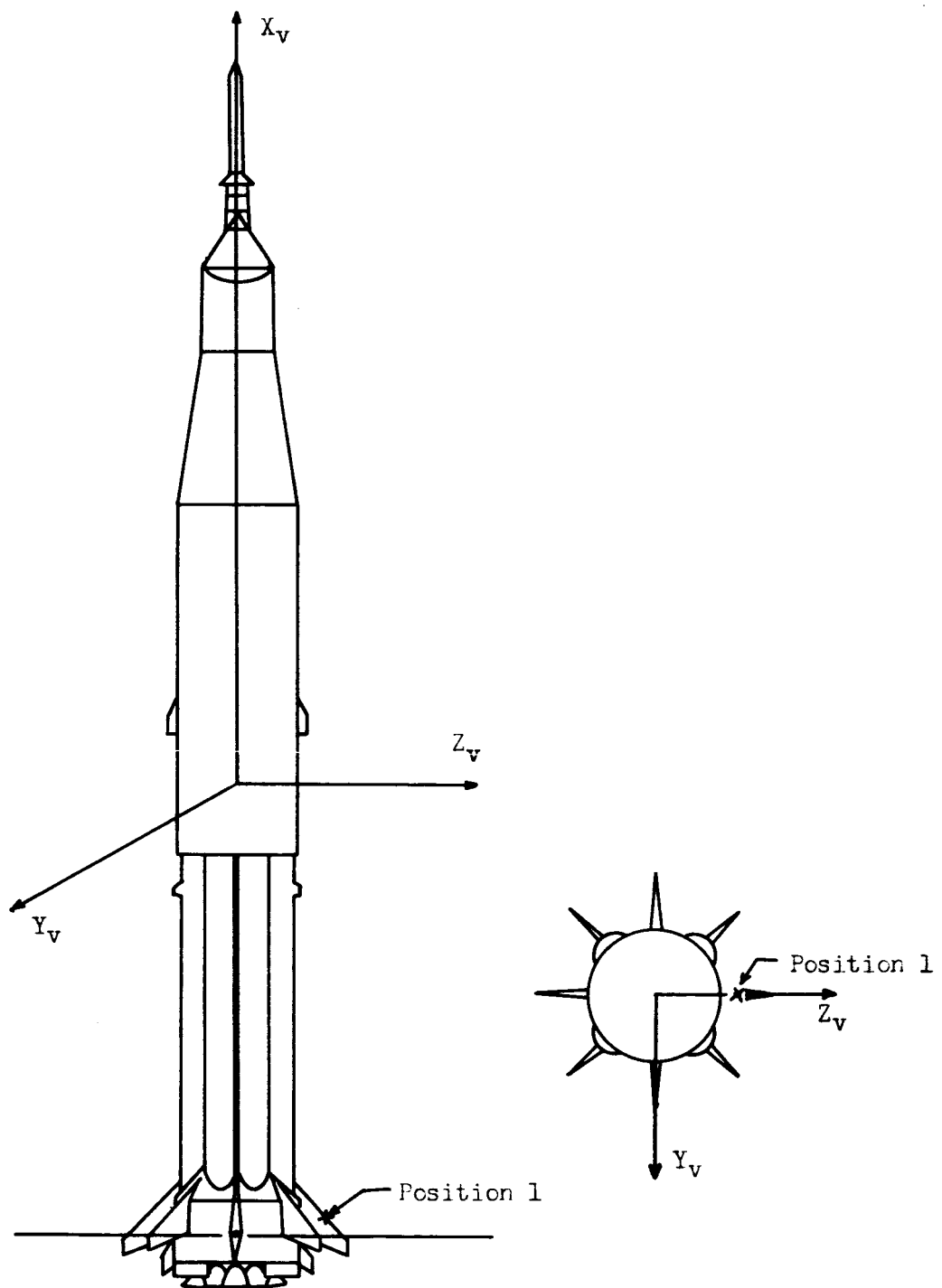


Fig. 1-2--Vehicle-Fixed Coordinate System.

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